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Solving univariate equations

Let's look back to the definition of *algebra*, a branch of mathematics that uses mathematical statements to describe relationships between things that vary. The mathematical statements used are *equations*, which means that two expressions are equivalent. In order to understand the relationship being represented, you will need to be able to solve the equations used to describe these relationships.

Suppose we are given the equation: $x + 3 = 15$. To ***solve the equation*** means to determine a numerical value for a variable (or variables) that makes this statement true. This is the number that, when added to 3, gives the result of 15. With this simple equation, you can see that the answer is 12. $12 + 3 = 15$.

In this unit, we will begin our discussion solving equations that contain single variables. We saw that the single variable equation $x + 3 = 15$ could be solved by inspection; but many single variable equations are far more complex than this. For these more complex equations, we will need to use systematic methods for finding solutions.

Steps for Solving Equations

1. Combine like terms.
2. Isolate the terms that contain the variable.
3. Isolate the variable you wish to solve for.
4. Substitute your answer into the original equation to see if it works.

To perform these steps, you will need to use a number of mathematical properties of addition, subtraction, multiplication and division.

The Addition Property of Equality and Its Inverse Property of Subtraction

If $a = b$, then $a + c = b + c$

If $a = b$, then $a - d = b - d$

In other words, adding the same quantity to both sides of an equation produces an equivalent equation. Since subtraction is simply adding a negative number, this rule applies when subtracting the same quantity from both sides.

Let's try this with the example: $x + 3 = 15$.

The key to solving this equation is to *isolate x.* On the left side of the equation, x is added to 3. To undo this addition we must subtract 3 from both sides of the equation. It is important that we subtract 3 from **both** sides of the equation; otherwise we will lose equality.

Subtract 3 from both sides.	$(x + 3) - 3 = 15 - 3$ $x = 12$

We can easily check the result we found above by substituting 12 for x into the original equation.

The 12 works in this equation, so the answer is correct.	$12 + 3 = 15$

The Multiplication Property of Equality and The Inverse Operation of Division

If $a = b$, then $ac = bc$ where c does not equal 0
If $c = d$, then $c/e = d/e$ where e does not equal 0

Multiplying both sides of an equation by the same non-zero number produces an equivalent equation. We may adapt this property to state that if we divide both sides of an equation by the same non-zero number, we obtain an equivalent equation.

This fact follows from knowing that multiplying by the reciprocal of a number is the same thing as dividing by that number.

$c \bullet 1/e$ is equivalent to $c \div e$

For example, suppose you want to solve the following equation for *x*:

$$3x = 72$$

To solve this problem, we want to *isolate* the *x* variable. Since the variable is multiplied by a numerical coefficient, we can't use addition or subtraction to do this.

Since *x* is multiplied by 3, we divide both sides of the equation by 3 to *isolate x*.

$$3x = 72$$
$$\frac{3x}{3} = \frac{72}{3}$$
$$x = 24$$

Again, what we do to one side of the equation, we must also do to the other side of the equation.

Now let's work through an example and solve a univarite equation.

Step 1: Combine Like Terms

As we learned in the last unit, ***like terms*** are *terms* that contain the same variable or group of variables raised to the same exponent, regardless of their numerical coefficient. Keeping in mind that an ***equation*** is a mathematical statement that two expressions are equal, in this step we will focus on combining like terms for the two expressions contained in an equation.

Since this unit deals only with equations containing a single variable, there are not many like terms to deal with. If we are given the equation $3z + 5 + 2z = 12 + 4z$, we need to first combine like terms in each expression of this equation.

The two expressions in this equation are:
$3z + 5 + 2z$ and $12 + 4z$.

$$3z + 5 + 2z = 12 + 4z$$

There are three terms that contain the variable *z*: 3*z*, 2*z*, and 4*z*. Combine 3*z* and 2z on the left side of the equation, then subtract 4*z* from both sides.

$$(3z + 2z) + 5 = 12 + 4z$$
$$5z + 5 - 4z = 12 + 4z - 4z$$
$$z + 5 = 12$$

Notice we chose to subtract $4z$ from both sides rather than $5z$. We chose to do this because consolidating in this manner left z positive. However, subtracting $5z$ from both sides would also be correct.

See how it works when we subtract $5z$ from both sides.	$5z + 5 - 5z = 12 + 4z - 5z$ $5 = 12 - z$

Step 2: Isolate the Terms that Contain the Variable

The main idea in solving equations is to *isolate* the variable you want to solve for. This means we want to get terms containing that variable on one side of the equation, with all other variables and constants "moved" to the opposite side of the equation. This section will address how we "move" terms from one side of an equation to another, in order to *isolate* a variable, using addition and its inverse property of subtraction.

The example we started in step one, $3z + 5 +2z = 12 + 4z$, is an example of an equation that contains more than one term with a variable. In step one, we combined all terms containing the variable z:

$(3z +2z) + 5 = 12 + 4z$
$5z + 5 - 4z = 12 + 4z - 4z$
$z + 5 = 12$

Now we want to isolate the terms that contain z.

Subtract 5 from both sides to isolate the z.	$z + 5 - 5 = 12 - 5$
This gives us our final result.	$z = 7$

Step 3: Isolate The Variable You Wish To Solve For

In the examples above, by isolating the terms containing the variable we wished to solve for, we were left with a term that had a numerical coefficient of one, so the variable was automatically isolated. However, if the variable does not have a coefficient of one, we will need to isolate the variable itself. When the variable we wish to isolate is either multiplied or divided by a numerical coefficient (or other variables) that is not equal to one, we need to use either multiplication or division to isolate the variable.

Step4: Substitute Your Answer into the Original Equation

Every answer should be checked to be sure it is correct. ***Substitution*** is a process of replacing variables with numbers or expressions.

After finding the solution for a variable, substitute the answer into the original equation to be sure the equality holds true.

To be sure our answer is correct, we can check it by substituting the solution back into the original equation, $3z + 5 + 2z = 12 + 4z$:

$$3(7) + 5 + 2(7) = 12 + 4(7)$$
$$21 + 5 + 14 = 12 + 28$$
$$40 = 40$$

Notice that the right and left sides are equal. Therefore, we have the correct solution. Now let's look at some examples of using addition and subtraction to solve equations.

Let's work through another example. Solve:

$$7x - 2 = 8 + 2x$$

1. Combine like terms.

In the equation above, there are two terms containing x. We need to first combine these terms. We do this by subtracting $2x$ from both sides.

$$7x - 2 - 2x = 8 + 2x - 2x$$
$$5x - 2 = 8$$

2. Isolate the terms that contain the variable you wish to solve for.

We *isolate* $5x$ by adding 2 to each side of the equation.

$$5x - 2 = 8$$
$$5x - 2 + 2 = 8 + 2$$
$$5x = 10$$

3. Isolate the variable you wish to solve for.

Since x is multiplied by 5, we use the inverse operation, division, to isolate x.

$$5x \div 5 = 10 \div 5$$
$$5x/5 = 10/5$$
$$x = 2$$

4. Substitute your answer into the original equation and check that it works.

When we substitute x= 2 into the original equation, we get 12= 12. Therefore, x = 2 is correct.

$$7x - 2 = + 2x$$
$$7(2) - 2 = 8 + 2(2)$$
$$14 - 2 = 8 + 4$$
$$12 = 12$$

Example

Solve the following equations.

1. $5x – 2 = 3x + 10$
2. $4(3x + 1) = 3x + 22$

Answers

Solve the following equations.

1. $5x – 2 = 3x + 10$ **$x = 6$**
2. $4(3x + 1) = 3x + 22$ **$x = 2$**

Solving for One Variable in a Multivariate Equation

Equations containing more than one variable are referred to as "multivariate" equations.

When faced with a multivariate equation, you may either wish to find a numeric value for each variable, or solve for one variable in terms of the other.

In this section, we will review how to solve for one variable in terms of others. This means we will isolate one variable and our result will contain other variables.

When we are given a multivariate equation where we want to solve for just one variable, we follow the same steps used for equations with one variable.

Steps for Solving Single Multivariate Equations

1. Combine like terms.
2. Isolate the terms that contain the variable you wish to solve for.
3. Isolate the variable you wish to solve for.
4. Substitute your answer into the original equation and check that it works.

Example

Given the equation $2(5x + z) = 30x + 3y + 10$, find the value of x in terms of y and z.

Answer

1. Combine like terms.

First we need to expand the equation by multiplying out the parentheses. Then we combine the x terms in the equation. To do this, it's easiest to subtract $10x$ from both sides, since this leaves us with a positive value of x.

2(5x + z)=30x + 3y + 10
10x + 2z = 30x + 3y + 10
10x - 10x + 2z = 30x -10x + 3y + 10
2z = 20x + 3y + 10

2. Isolate the terms that contain the variable you wish to solve for.

Now we want to *isolate* the term with the *x* variable. To do this we subtract 3y and 10 from both sides of the equation and combine terms.

$2z = 20x + 3y + 10$
$2z - 3y - 10 = 20x + 3y - 3y + 10 - 10$
$2z - 3y - 10 = 20x$

3. Isolate the variable you wish to solve for.

Since the term that contains *x* is multiplied by 20, we divide both sides of the equation by 20 to *isolate x.*

$$2z - 3y - 10 = 20x$$
$$\frac{2z - 3y - 10}{20} = x$$

4. Substitute your answer into the original equation and check that it works.

Given 2(5x + z) = 30x +3y +10

and $x = \frac{2z-3y-10}{20}$

$$2(5(\frac{2z-3y-10}{20}) + z) = 30(\frac{2z-3y-10}{20}) + 3y + 10$$
$$2(\frac{2z-3y-10}{4} + z) = (\frac{6z-9y-30}{2}) + 3y + 10$$
$$(\frac{2z-3y-10}{2} + 2z) = (\frac{6z-9y-30}{2}) + 3y + 10$$
$$z - 1\tfrac{1}{2}y - 5 + 2z = 3z - 4\tfrac{1}{2}y - 15 + 3y + 10$$
$$-1\tfrac{1}{2}y - 5 + 3z = 3z - 1\tfrac{1}{2}y - 5$$

When we substitute the value for Q back into our original equation, we find that both sides of the equation are equal. Therefore, this solution is correct.

Solving Systems of Multivariate Equations

In the last section, we reviewed how to deal with solving for one variable when given a single equation that contains more than one variable. Specifically, we solved for one variable **in terms of another variable** (or variables). When we have only one equation that contains more than one variable, we cannot find a numeric answer for any one of those variables. However, often you will have situation where you will have more than one variable and more than one equation. In this section, we will be dealing with multiple equations, which are what we call a system of equations.

There are two methods used for solving systems of equations:

- addition/subtraction method
- substitution method

Either method may be used to solve any problem, but depending on the given information, one method may be advantageous. You should learn to do both methods, since there may be times when one is easier to use than the other. We will give guidelines for helping you determine which method you wish to use.

When solving any system of equations, **there must be the same number of equations as variables**. If there are two variables, there must be two equations; three variables, three equations, etc.

NOTE: We will only refer to systems of equations with two equations. You can use the same process for systems with more equations, but this requires using the steps in an iterative process. Once you have done this technique with two equations, you can apply it to systems with more equations.

Solving Systems of Equations by Substitution

Below are two conditions that should be met if you want to use substitution as a method for solving a system of equations.

- **There must be the same number of equations as variables**. If there are two variables, there must be two equations; three variables, three equations, etc.
- **One of the equations can easily be solved for one variable**.

The use of substitution does not exclude addition as another logical and efficient method, but if one variable is easily solved for, the substitution method may be easier to use.

Substitution Method for Solving Systems of Equations with Two Equations

1. Choose one equation and isolate one variable; this equation will be considered the first equation.
2. Substitute the solution from step 1 into the second equation and solve for the variable in the equation.
3. Using the value found in step 2, substitute it into the first equation and solve for the second variable.
4. Substitute the values for both variables into both equations to show they are correct.

Let's take a closer look at using this method for solving systems of two equations. We are given the two equations:

(a) $y - 3x = 5$ and
(b) $y + x = 3$

We want to determine if we can find a numerical value for both x and y.

1. Choose one equation and isolate one variable; this equation will be considered the first equation.

Look at equation (b). To isolate the y variable, we subtract x from both sides.

$$y + x = 3$$
$$y + x - x = 3 - x$$
$$y = 3 - x$$

2. Substitute the solution from step 1 into the second equation and solve for the variable in the equation.

In equation (a) $y - 3x = 5$, replace the variable y with the value for y obtained in step 1, from equation (b).

$$y - 3x = 5$$
$$(3 - x) - 3x = 5$$

Solve for x by combining like terms and *isolating* the terms with the x variable.

$$3 - x - 3x = 5$$
$$3 - 3 - 4x = 5 - 3$$
$$-4x = 2$$

Divide both sides by the coefficient of x to *isolate* x.

$$x = 2/-4 = -½$$
$$x = -½$$

3. Using the value found in step 2, substitute it into the first equation and solve for the second variable.

Now we take the value of $x = -½$ and plug it back into $y + x = 3$ and solve for the value of y.

$$y + x = 3$$
$$y + (-½) = 3$$
$$y - ½ = 3$$
$$y - ½ + ½ = 3 + ½$$
$$y = 3½$$

4. Substitute the values for both variables into both equations to show they are correct.

Now we substitute the value of $x = -½$ and $y = 3½$ into both of our original equations.

(a) $y - 3x = 5$	(b) $y + x = 3$
$3½ - 3(-½) = 5$	$3½ + (-½) = 3$
$3½ + 1½ = 5$	$3 = 3$
$5 = 5$	

Example

$D + Q = 57$
$00.10D + 0.25Q = 9.45$

Determine how many of the coins are quarters and how many are dimes.

Answer

1. Choose one equation and isolate one variable; this equation will be considered the first equation.

The equation $D + Q = 57$ can be easily solved for D.

We want to *isolate D*, so we subtract Q from both sides of the equation.

$D + Q = 57$
$D + Q - Q = 57 - Q$
$D = 57 - Q$

2. Substitute the solution from step 1 into the second equation and solve for the variable in the equation.

We substitute the value for D, which is $57 - Q$, into the other equation, $0.10D + 0.25Q = 9.45$.

$0.10D + 0.25Q = 9.45$, and
$D = 57 - Q$, so
$0.10\,(57 - Q) + 0.25Q = 9.45$

We find a numeric value for Q by *isolating Q*.

Get rid of the parenthesis using the distributive property. Combine like terms.

$5.7 - 0.10Q + 0.25Q = 9.45$
$5.7 + 0.15Q = 9.45$

Subtract 5.7 from both sides of the equation to isolate the term containing Q.

$5.7 + 0.15Q - 5.7 = 9.45 - 5.7$
$0.15Q = 3.75$

Divide both sides by 0.15.

$(0.15/0.15)Q = 3.75/0.15$
$Q = 25$

3. Using the value found in step 2, substitute it into the first equation and solve for the second variable.

We found that Q = 25, so we substitute that into the equation D + Q = 57.

$D + Q = 57$
$D + 25 = 57$
$D + 25 - 25 = 57 - 25$
$D = 32$

4. Substitute the values for both variables into both equations to show they are correct.

Substitute the value of $D = 32$ and $Q = 25$ into both of our original equations.

$D + Q = 57$	$0.10D + 0.25Q = 9.45$
$32 + 25 = 57$	$.10(32) + .25(25) = 9.45$
$57 = 57$	$3.20 + 6.25 = 9.45$
	$9.45 = 9.45$

These values work in both equations, so we have the correct answer.

Solving Systems of Equations by Addition/Subtraction

A second method that can be used to solve systems of equations is addition and subtraction. Below are two conditions that should be met if you want to use addition/subtraction for solving a system of equations.

- **There must be the same number of equations as variables.**
- **One variable from both equations has the same coefficient in both equations, or the coefficients are multiples of one another.**

For example, suppose we are given the following system of two equations:

$$2y - 6x = 4 \text{ and } 4y + 5x = 3$$

If we look at the term with the *y* variable in each equation, the numerical coefficients of both are multiples of 2. This means the addition/subtraction method could be used to solve this system of equations.

In many cases, this technique may be more efficient than using substitution. The method you choose to use is a matter of preference.

Addition/Subtraction Method for Solving Systems of Two Equations

1. Algebraically manipulate both equations so that all the variables are on one side of the equal sign and in the same order. (Line the equations up, one on top of the other.)
2. If needed, multiply one of the equations by a constant so that there is one variable in each equation that has the same coefficient.
3. Subtract one equation from the other.
4. Solve the resulting equation for the one variable.
5. Using the value found in the step 4, substitute it into either equation and solve for the remaining variable.
6. Substitute the values for both variables into the equation not used in step 5 to be sure the solution is correct.

Let's work through one example to see how the steps work. We have two equations below:

$$P = 2 + 2Q$$
$$P = 10 - 6Q$$

1. Algebraically manipulate both equations so that all the variables are on one side of the equal sign and in the same order. (Line the equations up, one on top of the other.)

   When we do this, we get the equations shown at the right. Notice how the terms containing P are lined up, as are the terms containing Q.

   $$P - 2Q = 2$$
   $$P + 6Q = 10$$

2. If needed, multiply one of the equations by a constant so that there is one variable in each equation that has the same coefficient.

The variable P in both equations has a coefficient of one, so we move on to the next step.

3. Subtract one equation from the other.

$$\begin{aligned}&(p - 2Q = 2)\\ &\underline{-(P + 6Q = 10)}\\ &(-2 - 6)Q = 2 - 10\\ &-8Q = -8\end{aligned}$$

4. Solve the resulting equation for the one variable.

$$\begin{aligned}&-8Q = -8\\ &\frac{-8Q}{-8} = \frac{-8}{-8}\\ &Q = 1\end{aligned}$$

5. Using the value found in the step 4, substitute it into either equation and solve for the remaining variable.

In the previous step we found that $Q = 1$. We plug this value back into either of our initial equations and solve for P.

$$\begin{aligned}P &= 2 + 2Q\\ P &= 2 + 2(1)\\ P &= 4\end{aligned}$$

6. Substitute the values for both variables into the equation not used in step 5 to be sure our solution is correct.

Now we substitute the value of $Q = 1$ and $P = 4$ into the equation not used in step 5.

$$\begin{aligned}P &= 10 - 6Q\\ 4 &= 10 - 6(1)\\ 4 &= 4.\end{aligned}$$

These values work in this equation, so we have the correct answer.

Keep in mind that you can use either method for solving systems of equations. Let's revisit the example we used in the last section; but this time, we will solve it using addition/subtraction.

Example

Solve the system of linear equations given below using addition/subtraction.

Suppose there is a piggybank that contains 57 coins, which are only quarters and dimes. The total number of coins in the bank is 57, and the total value of these coins is $9.45.

This information can be represented by the following system of equations:

$D + Q = 57$
$0.10D + 0.25Q = 9.45$

Determine how many of the coins are quarters and how many are dimes.

Answer
We were given the two equations:

1. $D + Q = 57$
2. $0.10D + 0.25Q = 9.45$

1. Algebraically manipulate both equations so that all the variables are on one side of the equal sign and in the same order.

When we do this, we get the equations shown at the right. Notice how the terms containing *D* are lined up, as are the terms containing *Q*.	D + Q = 57 0.10D + 0.25Q = 9.45

2. If needed, multiply one of the equations by a constant so that there is one variable in each equation that has the same coefficient.

Because neither of the variables has the same coefficient in both equations, we will need to multiply one equation by a number that will result in the variable in one equation having the same coefficient as in the other. In this example, we can see the coefficient of Q in equation (1) is 4 times larger than the coefficient of Q in equation (2).

If we multiply the second equation by 4, then the variable Q will have the same coefficient in both equations. This means that both sides of equation (2) must be multiplied by 4.	$0.10D + 0.25Q = 9.45$ $4(0.10D + 0.25Q = 9.45)$ $4(0.10)D + 4(0.25)Q = 4(9.45)$ $0.40D + 1.00Q = 37.8$

NOTE: We do this so that when we subtract one equation from the other, we are left with one variable.

3. Subtract one equation from the other.

$$\begin{array}{r} D + Q = 57 \\ \underline{- (0.40D + Q = 37.8)} \\ 0.60D = 19.2 \end{array}$$

4. Solve the resulting equation for the one variable.

We now solve the equation from step 3 to find a value for the *D* variable.

$$0.60D = 19.2$$
$$\frac{0.60D}{0.60} = \frac{19.2}{0.60}$$
$$D = 32$$

5. Using the value found in the step 4, substitute it into either equation and solve for the remaining variable.

Substitute the value of $D = 32$ into one of the equations to solve for Q.

$$D + Q = 57$$
$$32 + Q - 32 = 57 - 32$$
$$Q = 25$$

6. Substitute the values for both variables into the equation not used in step 5 to be sure our solution is correct.

Now we substitute the value of $D = 32$ and $Q = 25$ into the equation not used in step 5.

$$0.10D + 0.25Q = 9.45$$
$$.10(32) + .25(25) = 9.45$$
$$3.20 + 6.25 = 9.45$$
$$9.45 = 9.45$$

We find that $Q = 25$ and $D = 32$. If you look back at the last section, you will find this is the same result we found before.