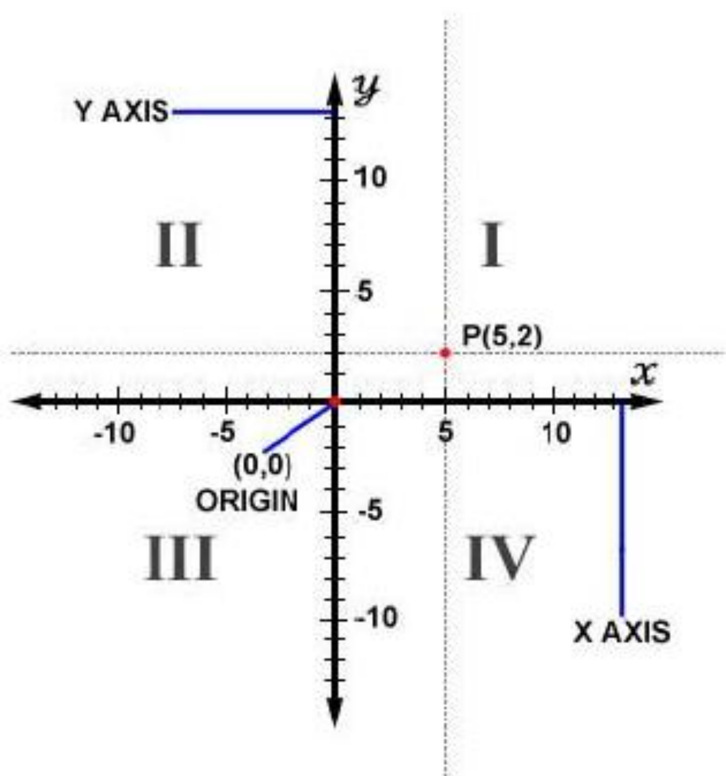
Click on the links below to jump directly to the relevant section

Elements of a graph

We often use graphs to give us a picture of the relationships between variables. Let's first look at the basic construction of graphs.

- _The modern Cartesian coordinate system in two dimensions (also called a **rectangular coordinate system**) consists of two axes called the *x* (horizontal) and *y* (vertical) axes. These axes correspond to the variables we are relating. We can also give the axes different names, such as Price and Quantity.
- _The point where the two axes intersect is called the *origin.* The origin is also identified as the point (0, 0).



2 DIMENSIONAL CARTESIAN COORDINATE SYSTEM

- _The arrows on the axes indicate that they extend forever in the same direction.
- _The intersection of the two axes creates four quadrants indicated by the roman numerals I, II, III, and IV. Conventionally, the quadrants are labeled counter-clockwise starting from the northeast quadrant. In Quadrant I the values are (x,y), and II:(-x,y), III:(-x,-y) and IV:(x,-y). (See table below.)

Quadrant	x values	y values
I	> 0	> 0
II	< 0	> 0
III	< 0	< 0
IV	> 0	< 0

- _To specify a particular point on a two dimensional coordinate system, you indicate the **x** unit first (abscissa), followed by the **y** unit (ordinate) in the form **(x,y)**. **(x,y)** is called an ordered pair.
- _An example of a point **P** on the system is indicated in the picture above using the coordinates **(5,2)**. 5 is the x-coordinate and 2 is the y-coordinate of point P.
- _**Note**: In a three dimensional coordinate system, another axis, normally labeled **z**, is added, providing a sense of a third dimension of space measurement. The axes are commonly defined as mutually orthogonal to each other (each at a right angle to the other). Coordinates in three dimensions are given as (x,y,z).

Coordinates of Points

A **coordinate** is one of a set of numbers used to identify the location of a point on a graph. Each point is identified by both an *x-coordinate* and a *y-coordinate*.

Identifying the x-coordinate

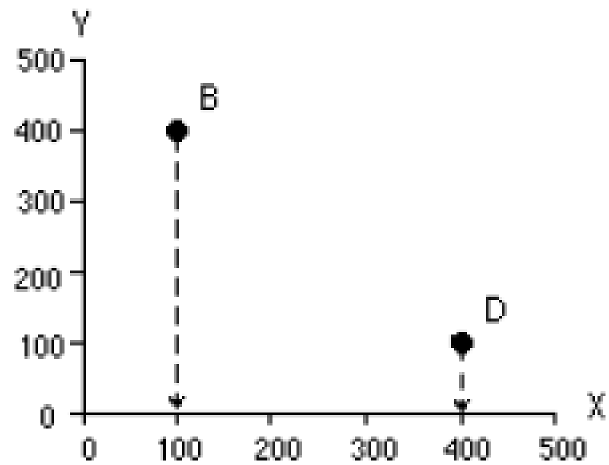
The ***x*-coordinate** of a point is the value that tells you how far from the origin the point is on the horizontal, or *x*-axis.

To find the *x*-coordinate of a point on a graph:

- _Draw a straight line from the point directly to the *x*-axis.
- _The number where the line hits the *x*-axis is the value of the *x*-coordinate.

Below is a graph with two points, B and D. In this figure:

- _The *x*-coordinate of point B is 100.
- _The *x*-coordinate of point D is 400.

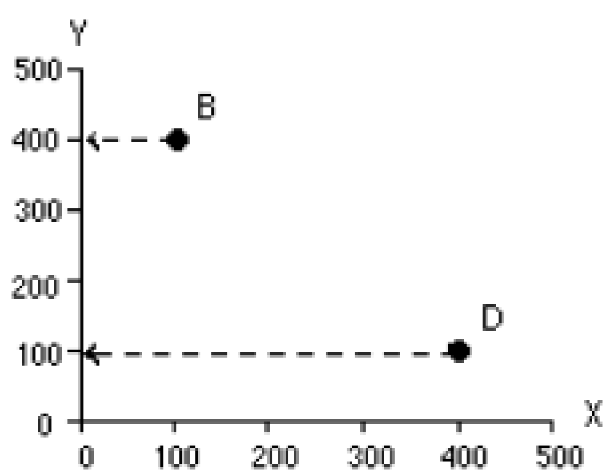


Identifying the y-coordinate
The ***y*-coordinate** of a point is the value that tells you how far from the origin the point is on the vertical, or *y*-axis. To find the *y*-coordinate of a point on a graph:

- _Draw a straight line from the point directly to the *y*-axis.
- _The number where the line hits the axis is the value of the *y*-coordinate.

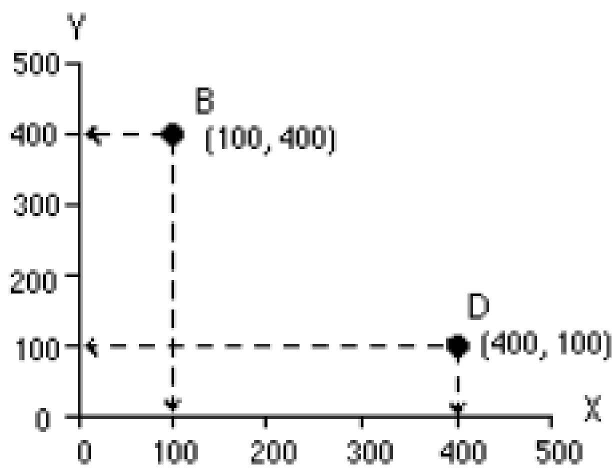
Looking back at the graph with our points B and D, we now identify the *y*-coordinate for each.

- _The *y*-coordinate of point B is 400.
- _The *y*-coordinate of point D is 100.



Notation for Identifying Points
Points are identified by stating their coordinates in the form of (x, y). Note that the x-coordinate *always* comes first.

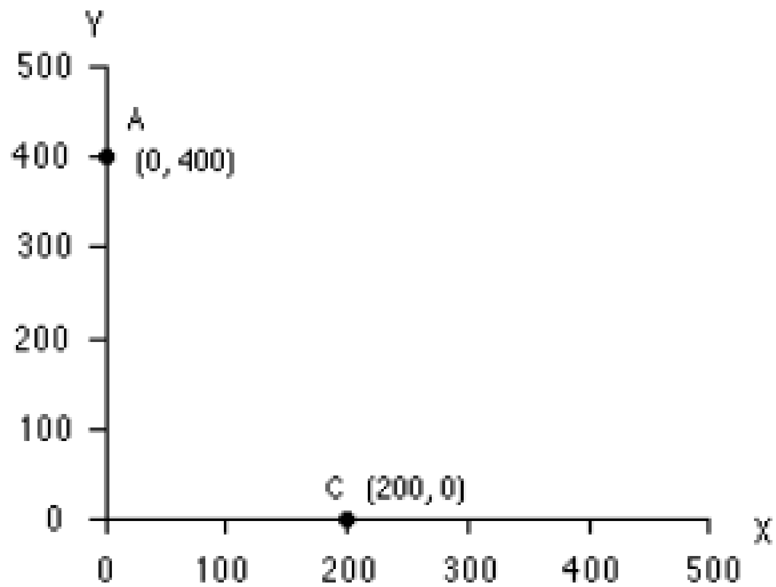
- _The x-coordinate of point B is 100.
- _The y-coordinate of point B is 400.
- _Coordinates of point B are (100, 400)
- _The x-coordinate of point D is 400.
- _The y-coordinate of point D is 100.
- _Coordinates of point D are (400, 100)



Points On The Axes
In the figure below, point A lies on the y-axis and point C lies on the x-axis. When a point lies on an axis, one of its coordinates must be zero.

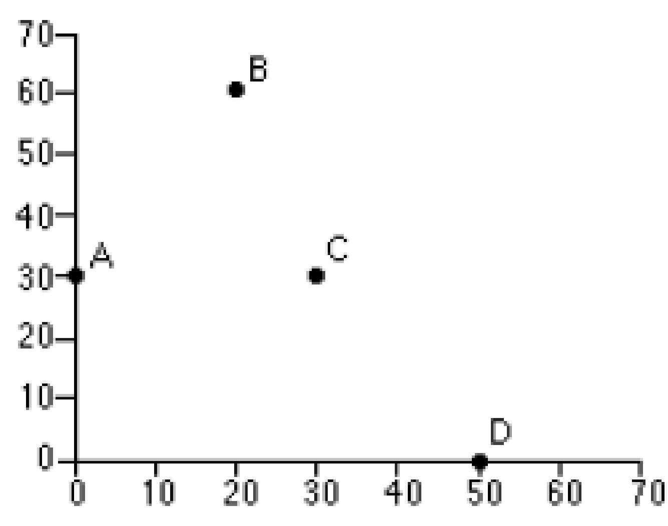
Point A--If you look at how far the point is from the origin along the x-axis, the answer is zero. Therefore, the x-coordinate is zero. Any point that lies on the y-axis has an x-coordinate of zero. If you move along the y-axis to find the y-coordinate, the point is 400 from the origin. The coordinates of point A are (0, 400).

Point C--If you look at how far the point is from the origin along the y-axis, the answer is zero. Therefore, the y-coordinate is zero. Any point that lies on the x-axis has a y-coordinate of zero. If you move along the x-axis to find the x-coordinate, the point is 200 from the origin. The coordinates of point C are (200, 0).



Example

1. Which point is labeled (20, 60)?
2. Which point(s) have a *y*-coordinate of 30?



Answers to Example

1. Which point is labeled (20, 60)? Point B
2. Which point(s) have a *y*-coordinate of 30? Points A & C

Plotting Points on a Graph

There are times when you are given a point and will need to find its location on a graph. This process is often referred to as plotting a point and uses the same skills as identifying the coordinates of a point on a graph. The process for plotting a point is shown using an example.

Example

Plot the point (200, 300).

Step One

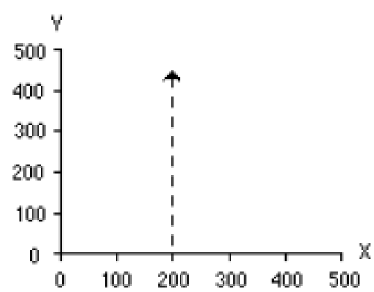
First, draw a line extending out from the *x*-axis at the *x*-coordinate of the point. In our example, this is at 200.

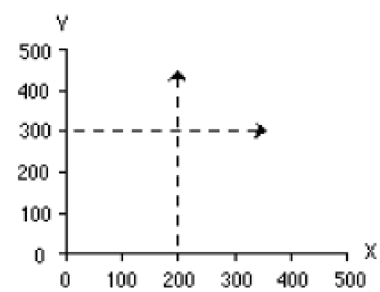
Step Two

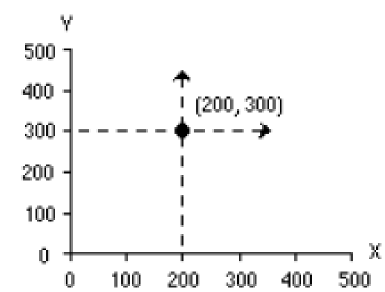
Then, draw a line extending out from the *y*-axis at the *y*-coordinate of the point. In our example, this is at 300.

Step Three

The point where these two lines intersect is at the point we are plotting, (200, 300).





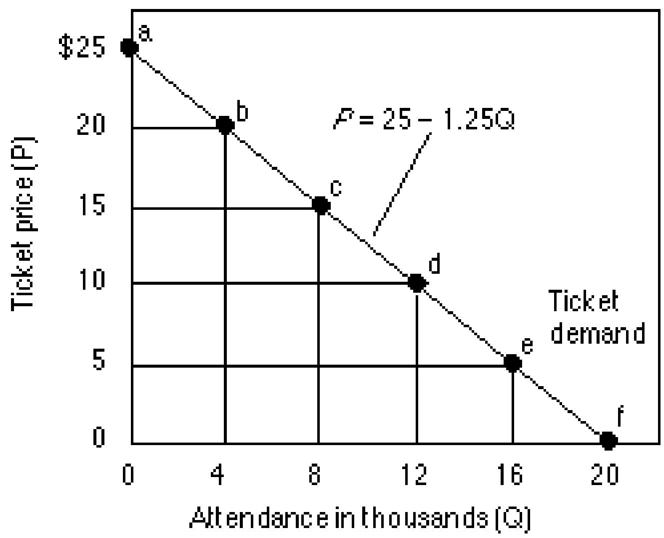


<u>Linear equations and their graphs</u>

Linear Relationships

Many graphs will display linear relationships, and you will need to interpret what is happening. There are several components of relationships that can be quickly determined from a graph once you know what to look for.

Two sets of data that are negatively or inversely related, such as ticket price and the attendance at basketball games, graph as a downward sloping line.



The example above shows the relationship between ticket prices and attendance. From this graph, you can see that this is a linear relationship where attendance will go down as ticket prices go up. This graph allows you to determine how much you should charge for a ticket. Using this graph involves an understanding of linear relationships and how to graph equations of straight lines.

Variables and Constants

You will often come across characteristics or elements such as rates, outputs, income, etc., measured by numerical values. Some of these will always remain the same, and some will change. The characteristic or element that remains the same is called a **constant.** For example, the number of donuts in a dozen is always 12. Therefore, the number of donuts in a dozen is a constant.

While some of these characteristics or elements remain the same, some of these values can vary (e.g., the price of a dozen donuts can change from $2.50 to $3.00). We call these characteristics or elements *variables*. **Variable** is the generic term for any characteristic or element that changes. You should be able to determine which characteristics or elements are constants and which are variables.

Relationships Between Variables

We express a relationship between two variables, which we will refer to as ***x*** and ***y***, by stating the following: The value of the variable ***y*** depends upon the value of the variable ***x***. We can write the relationship between variables in an equation. For instance:

$$\boldsymbol{y} = a + b\boldsymbol{x}$$

is an example of a relationship between *x* and *y* variables. The equation also has an "a" and "b" in it. These are constants that help define the relationship between the two variables.

- _In this equation the ***y*** variable is dependent on the values of ***x***, a, and b. The ***y*** is the **dependent variable**.
- _The value of ***x***, on the other hand, is independent of the values ***y***, a, and b. The ***x*** is the **independent variable.**

The following is an example to illustrate how these equations are constructed.

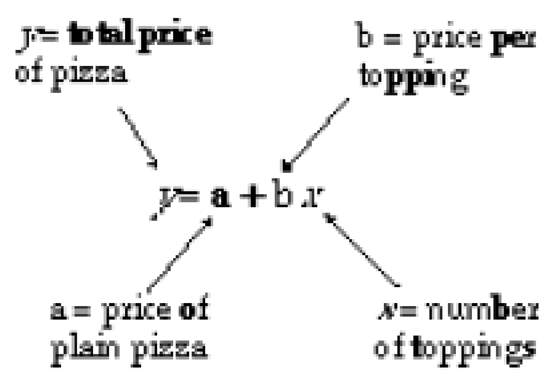
Throughout this tutorial we will use an example of a pizza shop that charges 7 dollars for a plain pizza with no toppings and 75 cents for each additional topping added. The total price of a pizza (***y***) depends upon the number of toppings (***x***) you order.

The price of a pizza is a dependent variable and number of toppings is the independent variable. In this example, both the price and the number of toppings can change; therefore, both are variables.

The total price of the pizza also depends on the price of a plain pizza and the price per topping. In our example, the price of a plain pizza and the price per topping do not change; therefore, these are constants. The relationship between the price of a pizza and the number of toppings can be expressed as an equation of the form: ***y*** = a + b***x***

If we know that ***x*** (the number of toppings) and ***y*** (the total price) represent variables, what are *a* and *b*? In our pizza example, "a" is the price of a plain pizza with no toppings and "b" is the price of each topping. They are constant. In other words, they are fixed values that specify how ***x*** relates to ***y***.

We can set up an equation to show how the total price of pizza relates to the number of toppings ordered.



If we create a table of this particular relationship between *x* and *y*, we'll see all the combinations of ***x*** and ***y*** that fit the equation. For example, if plain pizza (a) is $7.00 and price of each topping (b) is $.75, we get:

y = 7.00 + .75***x***

given:

$$y = a + b\,x$$

y Final Price	a Plain	b Price of Each Topping	*x* Number of Toppings
$ 7.00	$ 7.00	$.75	0
7.75	7.00	.75	1
8.50	7.00	.75	2
9.25	7.00	.75	3
10.00	7.00	.75	4

The procedure for graphing this relationship:

1. Generate a list of points for the relationship.
2. Draw a set of axes and define the scale.
3. Plot the points on the axes.
4. Draw the line by connecting the points.

1. **Generate a list of points for the relationship.** The first step in the process is to generate a list of points to graph. You do this by selecting several values for the *x*-coordinate. (**NOTE**: While two points define a straight line, using three points serves as an extra check that you have done all the calculations correctly. All three points should lie on the same straight line.) Once you have selected values for *x*, use the equation to calculate their corresponding *y* values.

   In the pizza example, the equation is $y = 7.00 + .75x$. You first select values of *x* you will solve for. You then substitute these values into the equation and solve for the *y* values.

   In the table below, for each given *x* value you can see the calculation of each of the *y* values.

	x Number of Toppings	y Final Cost
Let $x = 0$: $y = 7.00 + .75\ (0)$ $y = 7.00 + 0$ $y = 7.00$	0	7.00
Let $x = 1$: $y = 7.00 + .75\ (1)$ $y = 7.00 + .75$ $y = 7.75$	1	7.75
Let $x = 2$: $y = 7.00 + .75\ (2)$ $y = 7.00 + 1.50$ $y = 8.50$	2	8.50
Let $x = 3$: $y = 7.00 + .75\ (3)$ $y = 7.00 + 2.25$ $y = 9.25$	3	9.25
Let $x = 4$: $y = 7.00 + .75\ (4)$ $y = 7.00 + 3.00$ $y = 10.00$	4	10.00

After you have completed your table, you should end up with the following list of points:

(0, 7.00) , (1, 7.75), (2, 8.50), (3, 9.25), (4, 10.00)

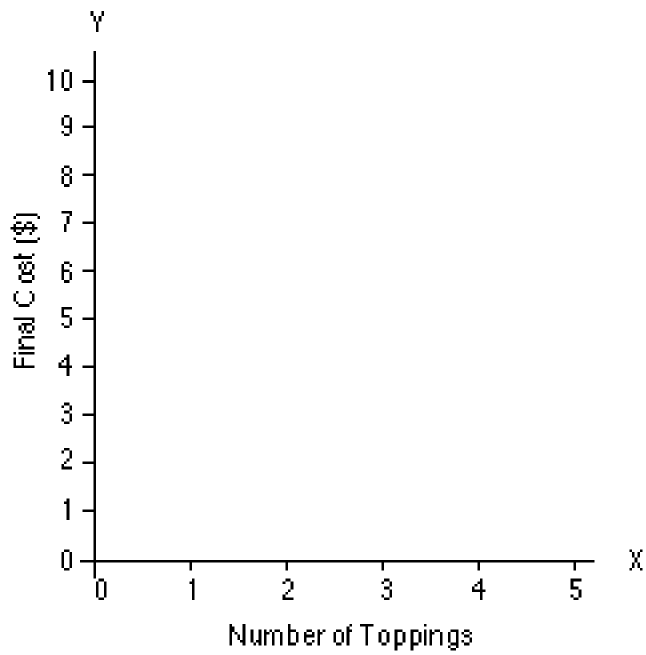
Notice the coordinates for each point are written using the (x, y) notation. You are now ready to create the graph of these points on a set of axes.

2. **Draw a set of axes and define the scale.**

Once you have your list of points, you are ready to plot them on a graph. The first step in drawing the graph is setting up the axes and determining the scale. The points you have to plot are:

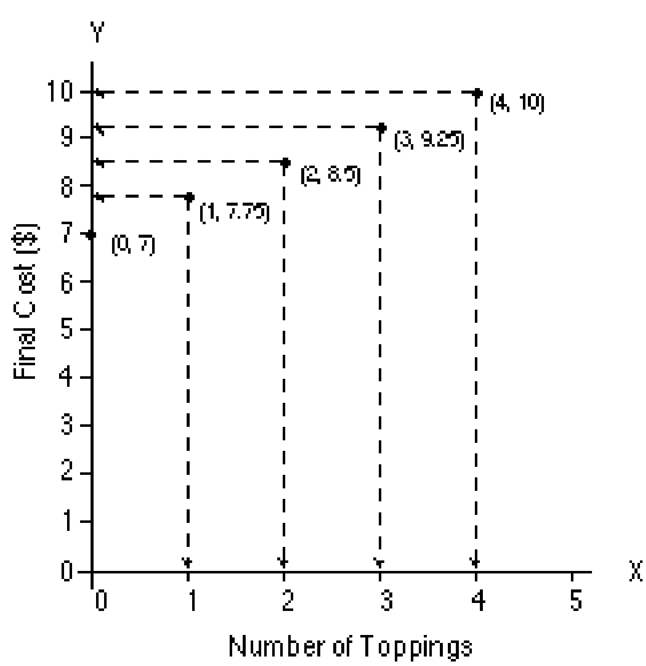
(0, 7.00), (1, 7.75), (2, 8.50), (3, 9.25), (4, 10.00)

Notice that the x values range from 0 to 4 and the y values go from 7 to 10. The scale of the two axes must include all the points. The distance between the points must be equal on each axis but does not have to be the same for both axes. On the diagram below, notice that the x-axis goes up to 5 and the y-axis goes up to 10; the scale on each axis can be different.



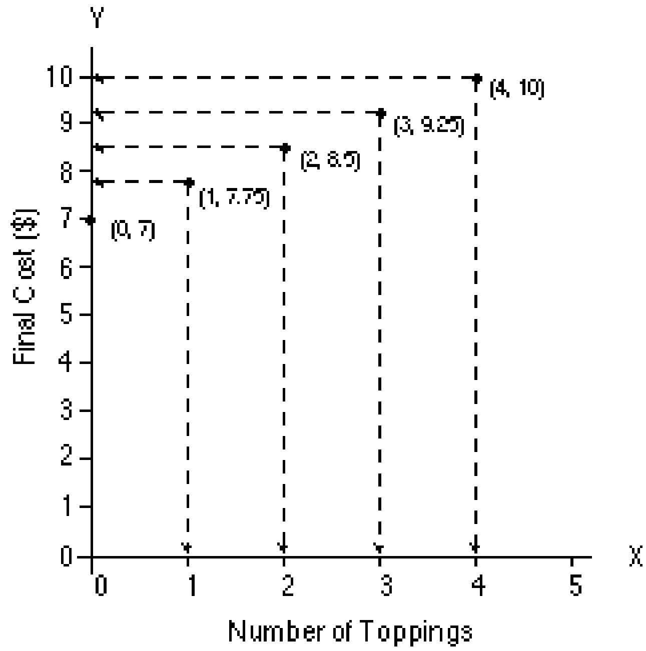
3. **Plot the points on the axes.**

After you have drawn the axes, you are ready to plot the points.



4. **Draw the line by connecting the points.**

Once you have plotted each of the points, you can connect them and draw a straight line.



Checking a Point in the Equation

If, by chance, you have a point and you wish to determine if it lies on the line, you simply go through the same process as generating points. Use the *x* value given in the point and insert it into the equation. Compare the *y* value calculated with the one given in the point.

Example

Does point (6, 10) lie on the line $y = 7.00 + .75\,x$ given in our pizza example? To determine this, we need to plug the point (6, 10) into the equation.

	x **Number of Toppings**	***y*** **Final Cost**
Let $x = 6$ $y = 7.00 + .75\ (6)$ $y = 7.00 + 4.50$ $y = 11.50$	6	11.50

As we can see here, the point with an *x* value of 6 that does lie on the line is (6, 11.5). This means that the point (6, 10) *does not* lie on our line.

What is slope?

The slope is used to tell us *how much* one variable (*y*) changes in relation to the change of another variable (*x*).

$$slope = \frac{\text{change in } y}{\text{change in } x}$$

We also refer to slope as a ratio of rise to run. Slope = rise/run.

Let's go back to the pizza example. As you may recall, a plain pizza with no toppings was priced at 7 dollars. As you add one topping, the cost goes up by 75 cents.

In the case of our pizza example, the slope is given in the equation on the right. The price of a pizza changes at a rate of 75 cents for each one topping added.

$$slope = \frac{\text{change in price}}{\text{change in number of toppings}}$$

$$slope = \frac{.75}{1} = .75$$

Calculating the Slope

To calculate the slope of a line you need only two points from that line, $(x1, y1)$ and $(x2, y2)$.

The equation used to calculate the slope from two points is:

$$slope = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Note: $(y_2$-$y_1)$ = rise $(x_2$-$x_1)$ = run Therefore, slope= rise/run.

There are three steps in calculating the slope of a straight line when you are not given its equation.

1. **Step One:** Identify two points on the line.
2. **Step Two:** Select one to be $(x1, y1)$ and the other to be $(x2, y2)$.
3. **Step Three:** Use the slope equation to calculate slope.

Rate of Change

The ratio of the change in quantity Y to a change in a related quantity, X, is called the **rate of change** of Y. We calculate the rate of change of Y by dividing the change in Y by the change in X.

The rate of change in Y =

$$\frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Now look at the equation for slope. The equation for slope is the same as the equation for calculating the rate of change. Slope represents a rate of change.

NOTE: Recall that the **growth rate (or percentage change)** was calculated by measuring the change in X from time one to time two, dividing the change by the value for X in time one, and multiplying by 100. **Growth rate** (or percentage change) is a measure of the change in a single variable, whereas **rate of change** measures the ratio of change between two related variables.

Example:

Let's say that points (15,8) and (10,7) lie on a straight line. What is the slope of this line?

1. **Step One:** Identify two points on the line.

We are given two points, (15, 8) and (10, 7), on a straight line.

2. **Step Two:** Select one to be ($x1, y1$) and the other to be ($x2, y2$).

It doesn't matter which we choose, so let's take (15, 8) to be ($x2, y2$). Let's take the point (10, 7) to be the point ($x1, y1$).

3. **Step Three:** Use the equation to calculate slope.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(8 - 7)}{(15 - 10)} = \frac{1}{5}$$

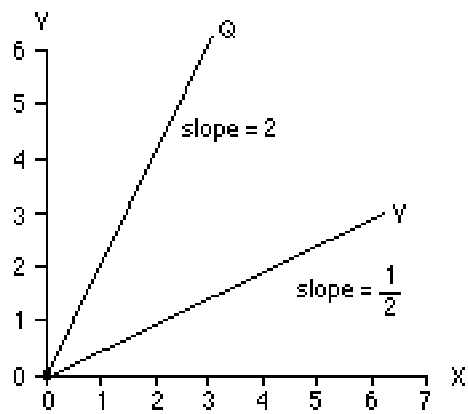
We said that it really doesn't matter which point we choose to be ($x1, y1$) and which point we choose to be ($x2, y2$). Let's show that this is true.

Take the same two points (15, 8) and (10, 7), but this time we will calculate the slope using (15, 8) as ($x1, y1$) and (10, 7) as the point ($x2, y2$). Then substitute these into the equation for slope:

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(7 - 8)}{(10 - 15)} = \frac{-1}{-5} = \frac{1}{5}$$

We get the same answer as before.

Now, take a moment to compare the two lines on the graph below. Notice that the line with the greater slope is the steeper of the two. The greater the slope, the steeper the line.



Determining Whether the Slope of a Line is Positive, Negative, Infinite or Zero

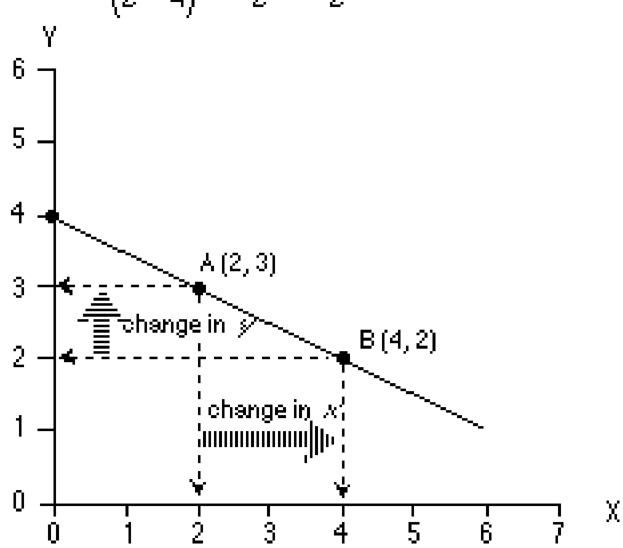
Let's take a moment to look at what happens when a line has a negative slope.

Keeping in mind that the slope is given by:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$

In the figure, the slope of the line is:

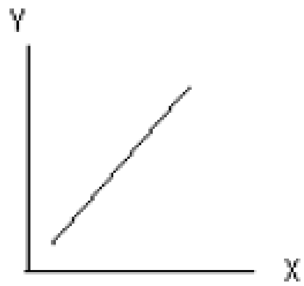
$$\text{slope} = \frac{(3-2)}{(2-4)} = \frac{1}{-2} = -\frac{1}{2}$$



The slope of this line is negative.

Pattern for Sign of Slope

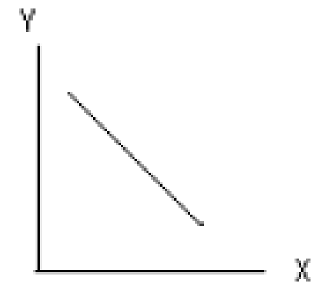
If the line is sloping upward from left to right, the slope is positive (+).



This means that *x* and *y* have a positive (direct) relationship. As *x* increases, *y* increases. *x* and *y* move in the same direction.

In our pizza example, a positive slope tells us that as the number of toppings we order (*x*) increases, the total cost of the pizza (*y*) also increases.

If the line is sloping downward from left to right, the slope is negative (-).

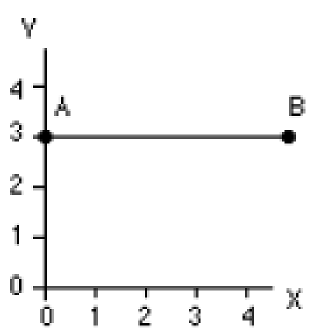


This means that *x* and *y* have a negative (inverse) relationship. As *x* increases, *y* decreases. *x* and *y* move in opposite directions.

For example, as the number of people that quit smoking (*x*) increases, the number of people contracting lung cancer (*y*) decreases. A graph of this relationship has a negative slope.

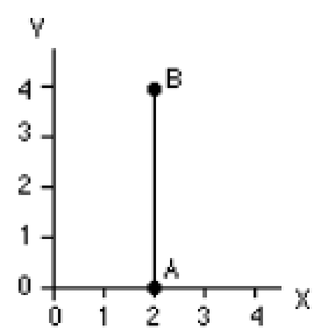
Two Other Cases to Consider

When the line is horizontal:



We can see that no matter what two points we choose, the value of the *y*-coordinate stays the same; it is always 3. Therefore, the change in *y* along the line is zero. No matter what the change in *x* along the line, the slope must always equal zero.

When the line is vertical:



In this case, no matter what two points we choose, the value of the *x*-coordinate stays the same; it is always 2. Therefore, the change in *x* along the line is zero.

$$\text{slope} = \frac{0}{\text{change in } x} = 0$$

Zero divided by any number is zero. **Horizontal lines have a slope of 0.**

$$\text{slope} = \frac{\text{change in } y}{0} = \infty$$

Since we cannot divide by zero, we say the slope of a vertical line is infinite. **Vertical lines have an infinite slope.**

Slope and y-intercept in the equation of a line

Equation of a Line The equation of a straight line is given on the right. In this equation:

- _"b" is the slope of the line
- _"a" is the *y*-intercept

variable slope

$$y = a + bx$$

y-intercept variable

Each of these will be defined below.

NOTE: The equation of a line may also be given as y = $\beta_0 + \beta_1 x$. In this form, β_0 (beta sub-zero) is the y-intercept and β_1 is the slope.
In general, the coefficient of *x* will be the slope whether it is b, m, or β_1.

Equation for Pizza Example Let's label the equation for our pizza example. The slope of the line tells us how much the cost of a pizza changes as the number of toppings change.

total cost of a pizza cost per topping

$$y = a + bx$$

price of plain pizza number of toppings

Slope of Pizza Example

As you found earlier, the equation for our pizza example is:

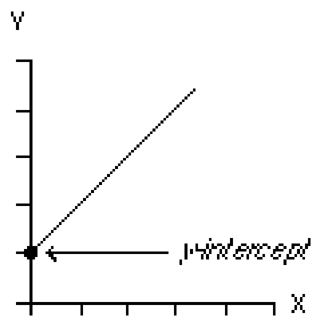
$$y = 7.00 + .75\,x$$

The slope in our pizza example is .75 (75 cents). In any equation of a straight line, the slope of the line is the constant that is multiplied by the *x* variable.

***y*-intercept of Pizza Example**

In the equation y = a + b x, the constant labeled "a" is called the *y-intercept*. The ***y*-intercept** is the point at which the line crosses the *y*-axis. The *y*-intercept is the value of *y* when *x* is equal to zero.

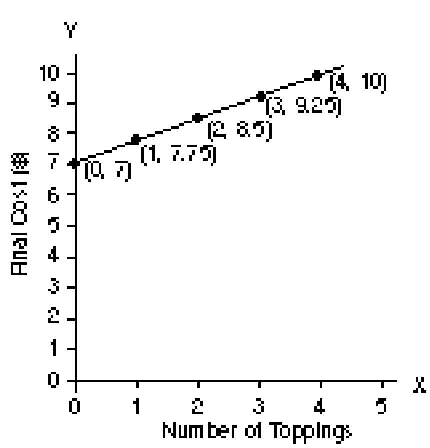
Note that if $x = 0$, then y = a. When we use graphs, we call this point (0, a) the y-intercept.



Note that if $x = 0$, then y = a. When we use graphs, we call this point (0, a) the y-intercept.

In our pizza example, the equation of the relationship is given by: $y = 7.00 + .75\,x$.

The *y*-intercept occurs when there are no additional toppings ($x = 0$), which is the price of a plain pizza, or $7.00. As you can see, the point where the line crosses the *y*-axis is (0, 7). The *y*-intercept in this case is 7.



Example

Determine the slope and y-intercept for the line given by each equation.

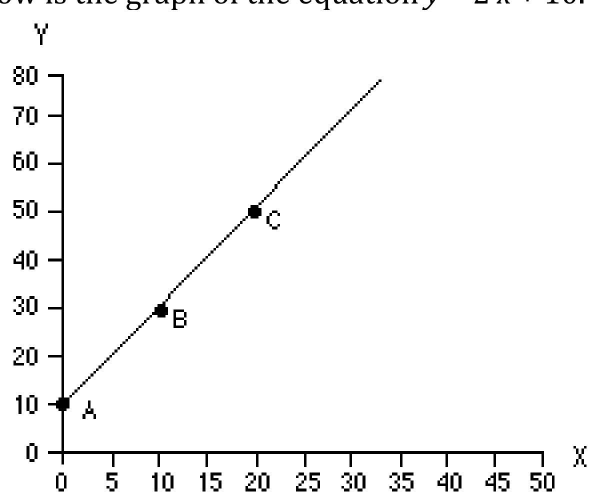
$y = 20 + 30x$	y -intercept is 20slope is 30
$y = 4 - 10x$	y -intercept is 4 slope is (-10)
$y = .5x + .66$	y -intercept is .66 slope is .5

Matching a Graph of a Straight Line with Its Equation

Another skill you will need is the ability to match an equation with its graph. One way to do this is to use the information you have about the equation of a straight line.

One way to match an equation of a straight line with the graph of a straight line is to use the slope and y-intercept.

Below is the graph of the equation $y = 2x + 10$.



We can check that this is the graph of the equation $y = 2x + 10$ by looking for two things:

• _Does the line cross the y-axis at 10?

• _Is the slope of the line on the graph 2?

By looking at the graph, you can see that the line does cross the y-axis at point A, (0, 10).

Now check for the slope. You can do this by using the points B, (10, 30), and C, (20, 50).

Using these points, the slope is:

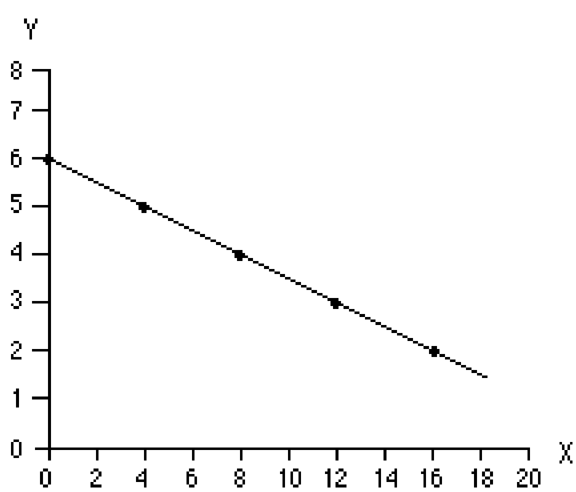
$$slope = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(50 - 30)}{(20 - 10)} = \frac{20}{10} = 2$$

Since the slope is found to be two, the graph of the line and the equation of the line match.

Example Consider the following graph at the right. Is the equation of the line shown in the graph below:

$y = 4 - 6x$, or

$y = 6 - 1/4x$?



The things we need to check for are:

- _The y-intercept
- _The slope of the line on the graph

  - _The y-intercept

    By examining the graph, you will notice that the line crosses the y-axis at the point (0, 6). Therefore, the y-intercept is 6.

  - _The slope of the line on the graph

Using the points (4, 5) and (12, 3) from the graph (NOTE: you can use any two points from the graph), the slope is calculated to be:

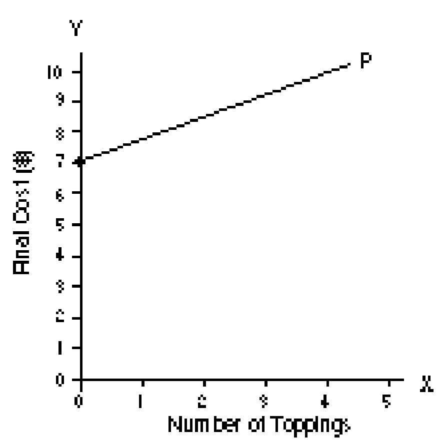
$$slope = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(3-5)}{(12-4)} = \frac{-2}{8} = -\frac{1}{4}$$

The slope of the line is -1/4.
The equation of the line is y= 6 - ¼ x.

Comparing lines on a graph

One skill that is extremely useful is the ability to draw conclusions about what is going on by examining graphs.

For example, the graph here shows the relationship between the number of toppings you select to put on a pizza and the final cost of a pizza (line P). Notice the line in this graph is labeled using a letter. This is one way to label a line.

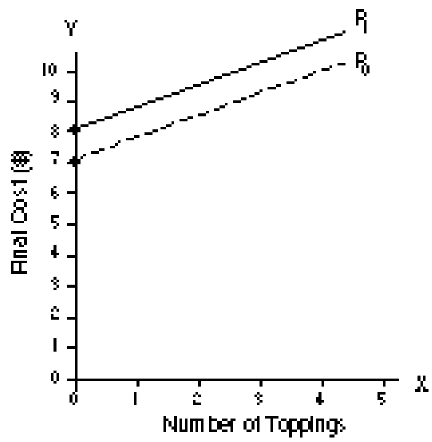


By looking at this graph, we can see that the cost of our plain pizza is $7.00, and the cost per topping is our slope, 75 cents.

This line has the equation of $y = 7.00 + .75x$.

Shift Due to Change in *y*-intercept

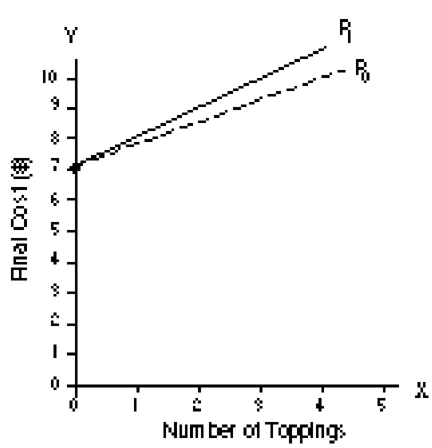
In the graph below, line P shifts from its initial position P0 to P1. When comparing the lines we find:



• _We can see the only change is that the line has shifted up.
• _The *y*-intercept, the point where the line crosses the *y*-axis, has changed from 7 to 8. Therefore, the initial price of the pizza has risen.
• _The equation for P0 is $y = 7.00 + .75x$, and the equation for P1 is $y = 8.00 + .75x$.

Shift Due to Change in Slope

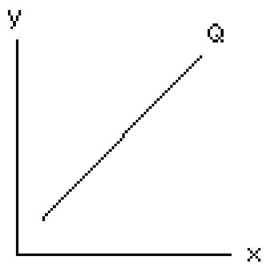
In the graph below, line P shifts from its initial position P0 to P1. When comparing the lines we find:

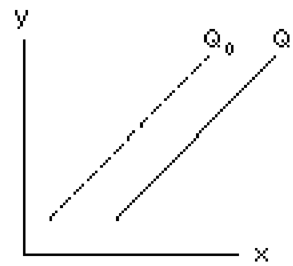


• _Both lines P0 and P1 still cross the *y*-axis at 7, but the line goes up at a different angle.
• _Line P1 is steeper than the line P0. This means that the slope of the equation has gone up. The slope has increased from .75 to 1. A plain pizza is still $7.00, but each additional topping now costs $1.
• _The equation for P0 is $y = 7.00 + .75x$, and the equation for P1 is $y = 7.00 + x$.

In general, when a line shifts in such a way that it maintains the same steepness as the original line (but moves up or down, or to the right or left), the *y*-intercept changes while the slope remains the same. If the line changes steepness, the slope must have changed.

Example
The straight line Q in the graph below is given by the equation y = t + px. If the line shifts from its initial position at Q0 to a the new position Q1, what must have changed in the equation?

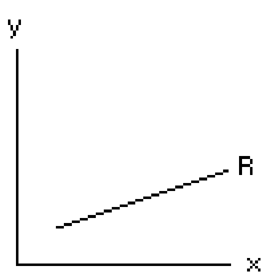


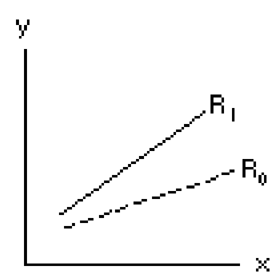


In this example, the line shifted to the right but did not change its slope. Therefore, only y-intercept, represented by the constant "t," changed in the equation.

If you extend both lines to intersect the y-axis, you will find Q1 intersects the y-axis at a smaller number, so "t" must have decreased. (**NOTE:** The line Q1 intersects the y-axis below the x-axis. This means that the y-intercept of Q1 is a negative number.) The constant "p," which is the slope, remains the same.

Example
The straight line R in the graph below is given by the equation y = Lx + Q. If the line shifts from its initial position at R0 to the new position R1, what must have changed in the equation?





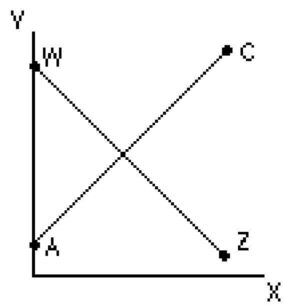
The line has changed in steepness. This means the slope (or the "L" constant) changed. Since R1 is steeper, "L" increased.

You may want to make some statement about the constant "Q." In a situation where we cannot see if the two lines both intercept the y-axis in the same place, it is hard to tell if "Q" changed or not. If you do extend both lines through the y-axis, you will find they have the same y-intercept, which means "Q" does not change. Unless we do this, we cannot make any statement about "Q" but can be certain that "L" does change.

Intersections and shifts of two lines

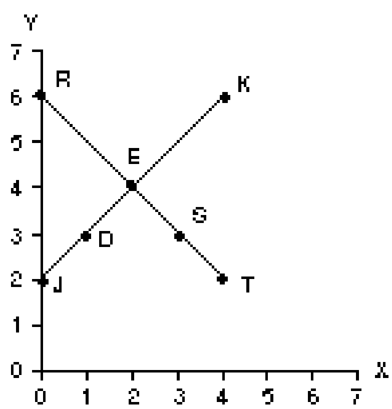
Many times we have the situation where there is more than one relationship between the x and y variables. You'll find this type of occurrence often in your study of supply and demand.

In this graph, there are two relationships between the *x* and *y* variables; one represented by the straight line AC and the other by straight line WZ.



Notice in this graph that each line is labeled by its endpoints. For example, the notation of AC refers to the line that has A and C as its endpoints. Using the endpoints is another way to label lines.

In the following graph, RT is downward sloping; it has a negative slope. Line JK is upward sloping; it has a positive slope.



Notice that there are several points on the graph. In all but one instance, the same *y* value corresponds to different *x* values on each line. For instance, each line in the graph has a point with a *y*-value of 3:

- _On the line RT, point S (3, 3) has a *y* value of 3 and an *x* value of 3.
- _On the line JK, point D (1, 3) has a *y* value of 3 and an *x* value of 1.

In one case, the two lines have the same (*x*, *y*) values simultaneously. This is where the two lines RT and JK **intersect**. The intersection occurs at point E, which has the coordinates (2, 4); the *x*-coordinate is 2, and the *y*-coordinate is 4.

Examining the Shift of a Line

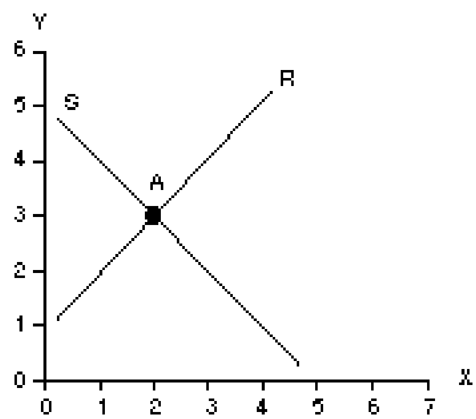
You will need to be able to draw conclusions from information presented in graphs where a line has shifted positions. It is important to be able to interpret what is happening on a graph after a shift occurs. You will typically be asked what happens to the intersection point of two lines when one of the lines shifts. In any situation where you are given a shift in a line:

• _identify both the initial and final points of intersection

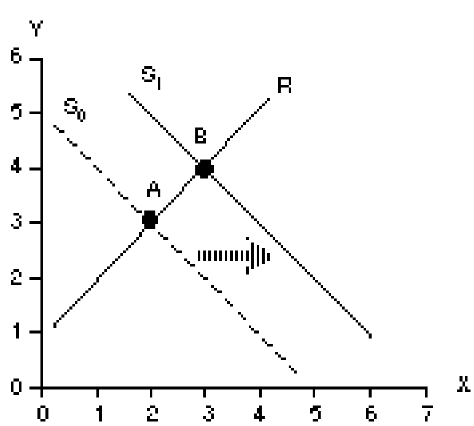
• _compare the coordinates of the two

\

Before the Shift
This graph contains the two lines *R* and *S*, which intersect at point *A* (2, 3). Line *S* shifts to the right. What happens to the intersection of the two lines if one of the lines shifts?

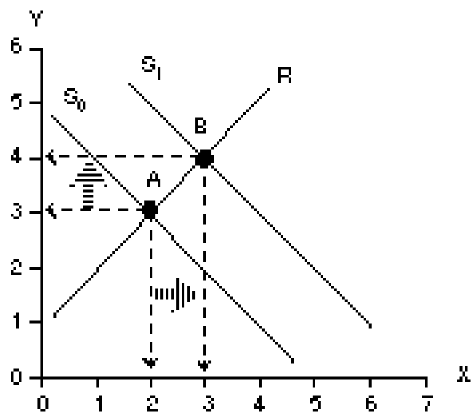


After the Shift
Before attempting to answer this question, we should look at a graph that illustrates this scenario. On the graph below, line *S0* is our original line *S*. Line *S1* represents our new *S* after it has shifted. The new point of intersection between *R* and *S* is now point *B* (3, 4).



Example

Compare the points A (2, 3) and B (3, 4) on this graph.



- _The *x*-coordinate changed from 2 to 3.
- _The *y*-coordinate changed from 3 to 4.

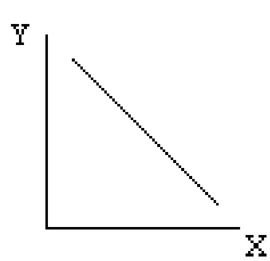
Both the *x-coordinate* and *y-coordinate* increased as the line S shifted to the right.

This is only one type of shift. S may shift to the left, R may shift to the right or left, or both lines may shift. The same procedure of comparing points of intersection can be used to examine any shift.

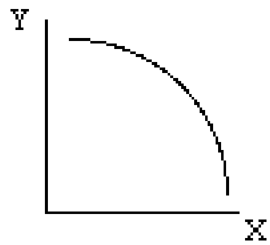
Nonlinear relationships

Many relationships are not linear. In such relationships, each unit change in the *x* variable will not always bring about the same change in the *y* variable. The graph of this relationship will be a curve instead of a straight line.

This graph below shows a **linear relationship** between *x* and *y*.



This graph below shows a **nonlinear relationship** between *x* and *y*.



Determining the Slope of a Curve at the Point of Tangency

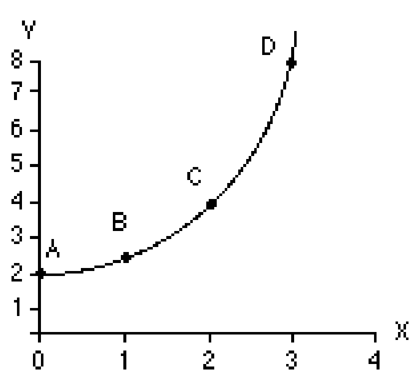
In the section on SLOPE, we looked at how to measure the slope of a straight line. Now we will examine how to find the slope of a point on a curve. One of the differences between the slope of a straight line and the slope of a curve is that the **slope of a straight line is constant, while the slope of a curve changes from point to point**.

As you recall, to find the slope of a line you need to:

1. **Step One:** Identify two points on the line.
2. **Step Two:** Select one to be $(x1, y1)$ and the other to be $(x2, y2)$.
3. **Step Three:** Use the slope equation to calculate slope.

$$\text{slope} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Now let's use the slope formula in a nonlinear relationship. Let's try using the procedure outlined above to find the slope of the curve shown below.



From point A (0, 2) to point B (1, 2.5)

$$\text{slope} = \frac{2.5 - 2}{1 - 0} = \frac{.5}{1} = \frac{1}{2}$$

From point B (1, 2.5) to point C (2, 4)

$$\text{slope} = \frac{4 - 2.5}{2 - 1} = \frac{1.5}{1} = 1\frac{1}{2}$$

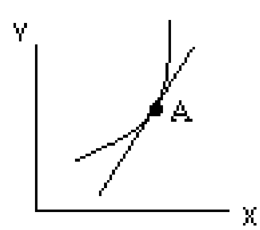
From point C (2, 4) to point D (3, 8)

$$\text{slope} = \frac{8 - 4}{3 - 2} = \frac{4}{1} = 4$$

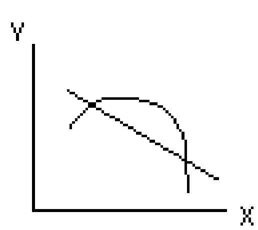
Here we see that the slope of the curve changes as you move along it. For this reason, we measure the slope of a curve at just one point. That is, instead of measuring the slope as the change between any two points (between A and B or B and C), we measure the slope of the curve at a single point (at A or C).

Tangent Line
To measure the slope of a curve at a single point, we must introduce the concept of a tangent. A tangent is a straight line that touches a curve at a single point and does not cross through it. The point where the curve and the tangent meet is called the point of tangency.

This curve has a tangent line to the curve with point A being the point of tangency. In this case, the slope of the tangent line is positive.

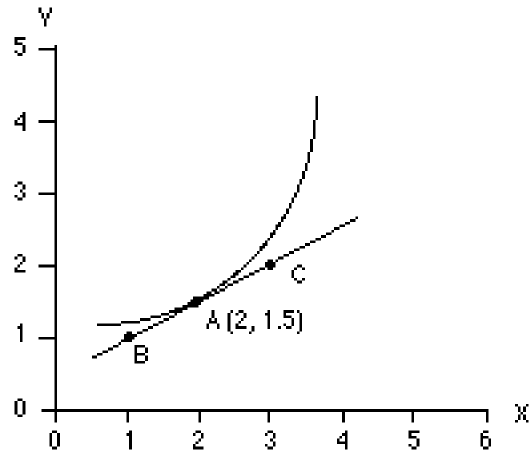


The line on this graph crosses the curve in two places. This line is **not** tangent to the curve.



The slope of a curve at a point is equal to the slope of the straight line that is tangent to the curve at that point.

Example: What is the slope of the curve at point A?



The slope of the curve at point A is equal to the slope of the straight line BC. By finding the slope of the straight line BC, we have found the slope of the curve at point A.

The slope of the line BC is ½, or 0.5. Threfore, the slope at point A is ½ or 0.5.

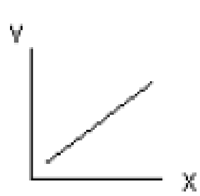
This is the slope of the curve **only at point A**. To find the slope of the curve at any other point, we would need to draw a tangent line at that point and then determine the slope of that tangent line.

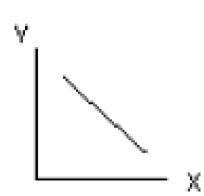
Determining Whether the Slope of a Curve is Positive, Negative, or Zero

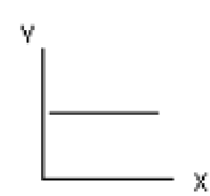
In the section on SLOPE, we made some generalizations concerning the slopes of straight lines.

The pattern for slope was:

If the line is sloping up to the right, the slope is positive (+).	If the line is sloping down to the right, the slope is negative (-).	Horizontal lines have a slope of zero (0).

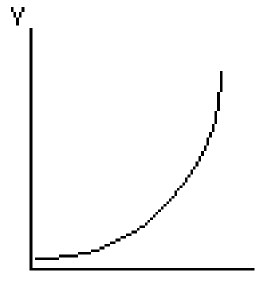


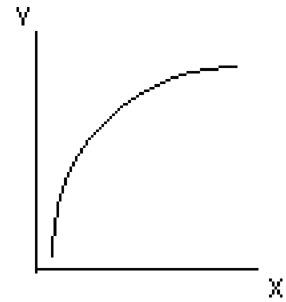




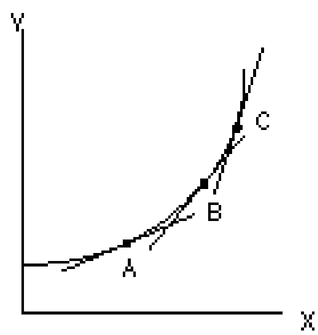
Curves with a Positive Slope

Both graphs below show curves sloping upward from left to right. As with upward sloping straight lines, we can say that generally the slope of the curve is positive. While the slope will differ at each point on the curve, it will always be positive.





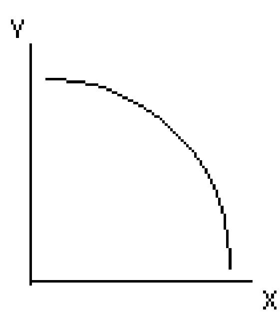
To check this, take any point on either curve and draw the tangent to the curve at that point.

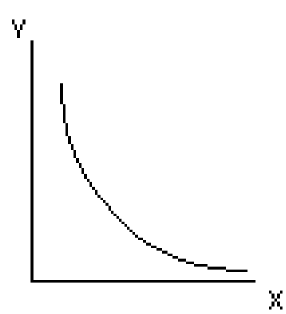


What is the slope of the tangent? The slope of the tangent is positive. For example, A, B, and C are three points on the curve. The tangent line at each of these points is different. However, each tangent has a positive slope; therefore, the curve has a positive slope at points A, B, and C. In fact, any tangent drawn to the curve will have a positive slope.

Curves with a Negative Slope

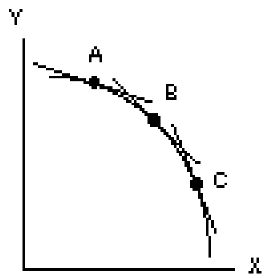
In the graphs below, both of the curves are downward sloping. Straight lines that are downward sloping have negative slopes; curves that are downward sloping also have negative slopes.





The slope changes from point to point on a curve, but all of the slopes along these two curves will be negative.

In general, to determine if the slope of a curve at any point is positive, negative or zero, you draw in the line of tangency at that point.

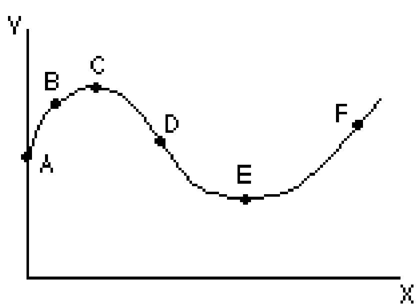


A, B, and C are three points on the curve. The tangent line at each of these points is different. However, each tangent has a negative slope (each is downward sloping). Therefore, the curve has a negative slope at points A, B, and C. All tangents to this curve will have negative slopes.

Example

In this example, our curve has a:

- _positive slope at points A, B, and F
- _a negative slope at D
- _a slope of 0 at points C and E (Remember, the slope of a horizontal line is zero.)

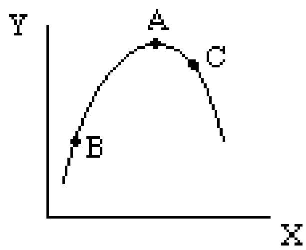


Maximum and Minimum Points of Curves

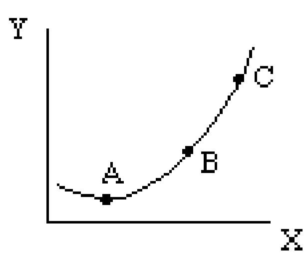
We can draw interesting conclusions from points on graphs where the highest or lowest values are observed. We refer to these points as maximum and minimum points.

- _Maximum and minimum points on a graph are found at points where the slope of the curve is zero.
- _The **maximum point** is the point on a curve with the highest *y*-coordinate and a slope of zero.
- _The **minimum point** is the point on a curve with the lowest *y*-coordinate and a slope of zero.

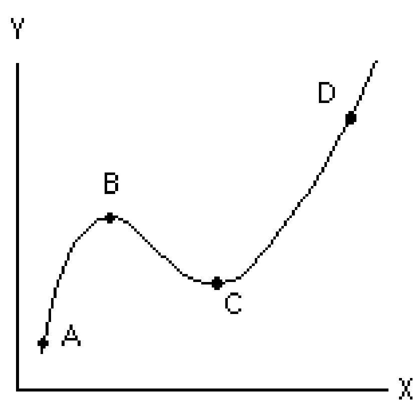
Maximum Point
Point A is at the maximum point for this curve. Point A is at the highest point on this curve. It has a greater y-coordinate value than any other point on the curve and has a slope of zero.



Minimum Point
Point A is at the minimum point for this curve. Point A is at the lowest point on this curve. It has a lower y-coordinate value than any other point on the curve and has a slope of zero.



Identify any maximum and minimum points on the curve.



• _The curve has a slope of zero at only two points, B and C.
• _Point B is the maximum. At this point, the curve has a slope of zero with the largest y-coordinate.
• _Point C is the minimum. At this point, the curve has a slope of zero with the smallest y-coordinate.
• _Point A clearly has the lowest y-coordinate of the points on the curve. Point D has the highest y-coordinate. However, at neither one of these points is the slope of the curve zero.