Basic Review

Components of a Fraction

A fraction is a number that is written in the form:

$$\text{or } a/b$$

- _a is the numerator and b is the denominator

Fractions are used when representing numbers that describe the parts of a whole. The fraction a/b also can be read as "a out of b," "a over b," or "a divided by b."

There are some restrictions on a and b.

- _Both a and b must be integers, meaning positive and negative whole numbers.

- _The denominator, or b, cannot be zero. This is because one cannot divide by zero.

Often you will read information that could be represented as a fraction. Below is an example of this type of information.

Example

If there are 18 students in a classroom, and 6 of the students wear glasses, what fraction of the students wear glasses? A fraction can be thought of as "a out of b." In the following picture:

- _The total number of students is 18.
- _The number of students with glasses is 6.
- _The number of students with glasses out of the whole class is 6/18 (or 1/3).

6/18
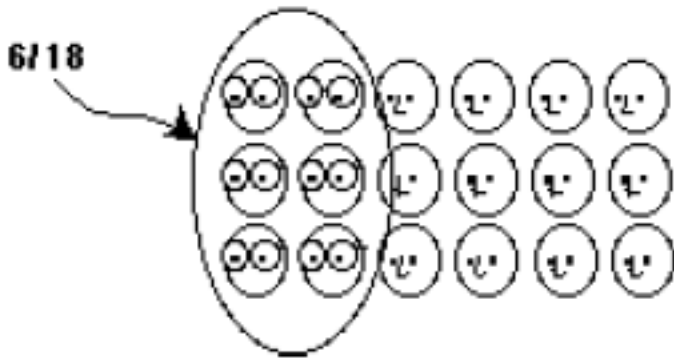
## *Comparing Fractions with Like Denominators or Like Numerators*

Comparing fractions gives you a sense of how items relate. Let's take a look at some fractions and see how changing the numerator, and then the denominator, changes a fraction.

Example

In each of the diagrams below, the shaded part represents the fraction shown below each rectangle.

The first rectangle has 1/1 (the whole rectangle) shaded. As you move down:
• _The numerator (top) of each fraction remains the same, the integer 1.
• _The denominator (bottom) increases.
• _As the **denominator gets larger**, the **shaded fraction gets smaller**.

$\frac{1}{1}$

$\frac{1}{2}$

$\frac{1}{3}$

$\frac{1}{4}$
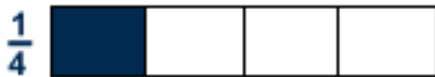
The first rectangle has 1/8 shaded. As you move down:
• _The numerator (top) of each fraction increases.
• _The denominator (bottom) remains the same, the integer 8.
• _As the **numerator gets larger**, the **shaded fraction gets larger**.

$\frac{1}{8}$

$\frac{2}{8}$

$\frac{3}{8}$

$\frac{4}{8}$
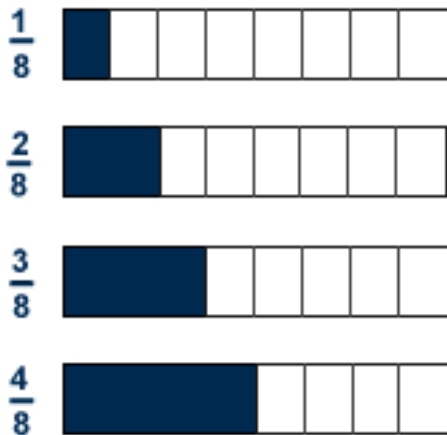
We can draw the following conclusions:

- When the numerator stays the same, and the denominator increases, the value of the fraction decreases.

- When the denominator stays the same, and the numerator increases, the value of the fraction increases.

Equivalent Fractions

Equivalent fractions are fractions that *may* look different but are equal to each other. Two equivalent fractions may have a different numerator and a different denominator.

$$\frac{a}{b} = \frac{c}{d}$$

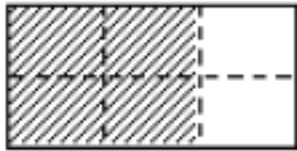Let's take a moment to demonstrate the concept of equivalent fractions.

1. Take a sheet of paper and fold it twice, creating three equal sections.

2. Now shade two of them. This shaded portion represents 2/3.

3. Fold the paper again, in the other direction, but down the center of the paper. The shaded portion is now 4/6.

The shaded portion of the paper does not change, so the fraction of the paper shaded does not change. The fractions 2/3 and 4/6 are equivalent.

$$\frac{2}{3} = \frac{4}{6}$$

If you multiply both the numerator and denominator of a fraction by the same non-zero number, the fraction remains unchanged in value. Therefore, equivalent fractions can be created by multiplying (or dividing) the numerator and denominator by the same number. This number is referred to as a multiplier.

In the demonstration above, we could get the fraction 4/6 by multiplying both the top and bottom of 2/3 by 2.

$$\frac{2 \times 2}{3 \times 2} = \frac{4}{6}$$

Example

Show that the fraction 8/12 is equivalent to the fraction 2/3. If you multiply both the numerator and denominator of 2/3 by 4, you get the fraction 8/12. Therefore, the two are equivalent.

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$$

Likewise, if you divide both the numerator and denominator of 8/12 by 4, you get the fraction 2/3. Therefore, the two are equivalent.

$$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

Writing Fractions in Simplest Form

Fractions can be simplified when the numerator and denominator have a common factor in them. If both the numerator and denominator have common factors, then we can cancel these factors out. For example, in the fraction 8/12, 4 is a common factor of both 8 and 12.

$$\frac{8}{12} = \frac{2 \times 4}{3 \times 4}$$

We can simplify the fraction by canceling the 4 from both the numerator and denominator of the fraction. Canceling is equivalent to dividing both the numerator and denominator by the same number.

$$\frac{8}{12} = \frac{2 \times 4}{3 \times 4} = \frac{2}{3} \quad \text{is the same as} \quad \frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}$$

The key to simplifying a fraction is to find all the common factors between the numerator and denominator and to eliminate them. The easiest way to be sure you have eliminated all the common factors of the numerator and denominator is to find the prime factors of each and then cancel them out.

Fundamental Theorem of Arithmetic Every composite number can be expressed as a product of prime numbers. These are referred to as prime factors.

- **_A factor of a number is a number that can be divided into the original number evenly (meaning there is no remainder).** For example, 4 is a factor of 8. That means 8 can be divided by 4 and there is no remainder (8 ÷ 4 = 2). This means that 2 is also a factor of 8.

- **_A prime number is a number that has only two factors, 1 and itself.** For example, the number 2 can be divided evenly only by itself and 1. Therefore, 2 is a prime number. The five smallest prime numbers are 2, 3, 5, 7, 11, and 13.

- **_Numbers that are not prime numbers are referred to as composite numbers.** The number 8 is a composite number since it has factors of 2 and 4 (and 1).

- **_1 is a factor of every number.**

To find the prime factors of a composite number, divide out prime numbers. For example, if we wish to find the prime factors of 24, we can start by dividing 24 by 2:

When we divide 24 by 2 we get a result of 12. Since 2 goes into 24 evenly, it is a factor of 24. 2 is also a prime factor since it can only be divided by itself and 1. Therefore, 24 has 2 as a prime factor and 12 as a composite factor.

$$24 \div 2 = 12$$

If we want to find all the prime factors of 24, we must continue by finding the factors of 12.

We can also divide 12 by 2. When we do this we find that 2 and 6 are factors of 12, with 6 being another composite factor.

$$6 \div 2 = 3$$

When we further divide 6 by 2 we get a result of 3. Finally, we have a result that is a prime number. Both 2 and 3 are prime factors of 6.

$$6 \div 2 = 3$$

Now we can look at our sequence of division and list all the prime factors of 24. To review our sequence of division we have:

$$24 \div 2 = 12$$
$$12 \div 2 = 6$$
$$6 \div 2 = 3$$

When we report the prime factors of 24, we must list each occurrence of a number. From this we can see that 2, 2, 2, and 3 are all the prime factors of 24.
A number can be written as the product of its prime factors, so 2 x 2 x 2 x 3= 24.

To simplify a fraction, follow four steps:

1. Write the prime factorization of both the numerator and denominator. (The process for finding prime factors was explained above).

2. Rewrite the fraction so that the numerator and denominator are written as the product of their prime factors.

3. Cancel out any common prime factors.

4. Multiply together any remaining factors in the numerator and denominator.

Example

Find the simplest form of the fraction 10/24.

  1. Write the prime factorization of both the numerator and denominator.

- The prime factors of 10 are 2 and 5.
- The prime factors of 24 are 2, 2, 2, and 3.

  2. Rewrite the fraction so that the numerator and denominator are written as the product of their prime factors.

$$\frac{10}{24} = \frac{2 \times 5}{2 \times 2 \times 2 \times 3}$$

  3. Cancel out any common prime factors. We can cancel out a 2 from both the numerator and the denominator.

$$\frac{10}{24} = \frac{2 \times 5}{\cancel{2} \times 2 \times 2 \times 3} = \frac{5}{2 \times 2 \times 3}$$

  4. Multiply together any remaining factors in the numerator and denominator.

$$\frac{10}{24} = \frac{5}{2 \times 2 \times 3} = \frac{5}{12}$$

The fraction 10/24 can be simplified to 5/12.

As we can see, finding prime factors is important for simplifying fractions. Once we find the prime factors, it is merely a matter of canceling out common prime factors.

Comparing Fractions

Review of Notation

x > y	x is greater than y	x $\geq$ y	x is greater than or equal to y
x < y	x is less than y	x $\leq$ y	x is less than or equal to y

Comparing fractions when the denominators are the same

When the denominators are the same, comparing fractions is easy. We simply compare the numerators. For example, ¾ > ¼ because 3 > 1.

Comparing fractions when the denominators are different

To compare fractions that have different denominators, convert them all to a set of fractions that have the same denominator. There are three steps to comparing fractions when the denominators are different.

1. Find the Least Common Denominator (LCD) for the group of fractions you are comparing.
2. Find the multiplier for each fraction. Multiply both the top and bottom of each fraction by that multiplier. The example below will detail how this is done.
3. Compare and order the numerators of each fraction.

Determine which is larger: 10/24 or 22/45.

1. Find the LCD for the group of fractions you are comparing.

When denominators are different, you can use equivalent fractions as a tool to create new fractions with the same denominator. This will make them easy to compare. This new denominator is called the least common denominator (LCD). **The least common denominator is the smallest number that is a common multiple of each of the original denominators.**

Finding the LCD of a Set of Fractions:
A. Write the prime factorization for the denominator of each fraction.
B. Note all prime factors that occur. For each prime factor that occurs, determine in which denominator it occurs the most. Write down the prime factor the number of times it occurs in that one denominator.
C. Calculate the LCD of your fractions. To do this, multiply the factors written down in step B.

To elaborate how these steps are done, let's find the LCD for 10/24 and 22/45:

A. Write the prime factorization for the denominator of each fraction.

- _Prime factors of 24 are: 2, 2, 2, and 3.
- _Prime factors of 45 are: 3, 3 and 5.

B. Note all prime factors that occur. For each prime factor that occurs, determine in which denominator it occurs the most. Write down the prime factor the number of times it occurs in that one denominator.

- _The prime factors that occur are 2, 2, 2, 3, 3, and 5.

NOTE: The prime factor 2 occurred only in 24, so we write that three times (the number of times 2 occurs in 24 as a prime factor). The prime factor 3   occurred in both 24 and 45 but **most often** in 45, so we write that two times   (the number of times 3 occurred in 45 as a prime factor). The prime factor 5 occurred only once (in 45), so we write that

once.

C. Calculate the LCD of your fractions. To do this, multiply the factors written down in step B.

$$2 \text{ x } 2 \text{ x } 2 \text{ x } 3 \text{ x } 3 \text{ x } 5 = 360$$

The LCD for these two fractions is 360.

2. Find the multiplier for each fraction. Multiply both the top and bottom of each fraction by that multiplier.

As you recall from above, to get the LCD we multiplied 2 x 2 x 2 x 3 x 3 x 5 = 360. Now look at 10/24 with its denominator written out as prime factors.

$$\frac{10}{24} = \frac{10}{2 \times 2 \times 2 \times 3}$$

If we compare the prime factors of 24 (2,2,2, and 3) to the prime factors of 360 (2,2,2,3,3, and 5), we notice that the factors missing are 3 and 5. Since 3 x 5 = 15, the multiplier that is missing is 15. This means we must multiply both the top and bottom of 10/24 by 15.

$$\frac{10}{24} = \frac{10 \times 15}{2 \times 2 \times 2 \times 3 \times 15} = \frac{150}{360}$$

Now do this for the fraction 22/45.

$$\frac{22}{45} = \frac{22}{3 \times 3 \times 5}$$

If we compare the prime factors of 45 (3,3, and 5) with the prime factors of 360 (2,2,2,3,3, and 5), we notice that the missing factors are 2, 2, and 2. Since 2 x 2 x 2 = 8, the multiplier that is missing is 8. We must multiply both the top and bottom of 22/45 by 8.

$$\frac{22}{45} = \frac{22 \times 8}{3 \times 3 \times 5 \times 8} = \frac{176}{360}$$

3. *Compare and order the numerators of each fraction.*

We can now see that the fractions we are comparing are 150/360 and 176/360. When we look at these two fractions:

$$\frac{176}{360} > \frac{150}{360} \quad \text{which means that} \quad \frac{22}{45} > \frac{10}{24}$$

Example

Order the following fractions from smallest to largest: 9/12, 14/18, 66/100.

1. Find the LCD for the group of fractions you are comparing. We found the LCD for these fractions in the last section. The LCD for our fractions is 2 x 2 x 3 x 3 x 5 x 5 = 900.

2. Find the multiplier for each fraction. Multiply both the top and bottom of each fraction by that multiplier.

Starting with 9/12: If we compare the prime factors of 12 (2,2, and 3) to the prime factors of 900 (2,2,3,3,5, and5), we are missing prime factors 3, 5, and 5. Since 3 x 5 x 5= 75, the multiplier for 9/12 is 75.

$$\frac{9}{12} = \frac{9}{2 \times 2 \times 3}$$

$$\frac{9}{12} = \frac{9 \times 75}{2 \times 2 \times 3 \times 75} = \frac{675}{900}$$

For 14/18: If we compare the prime factors of 18 (2,3, and 3) to the prime factors of 900 (2,2,3,3,5, and 5), we are missing prime factors 2, 5, and 5. Since 2 x 5 x 5 = 50, the multiplier for 14/18 is 50.

$$\frac{14}{18} = \frac{14}{2 \times 3 \times 3}$$

$$\frac{14}{18} = \frac{14 \times 50}{2 \times 3 \times 3 \times 50} = \frac{700}{900}$$

For 66/100: If we compare the prime factors of 100 (2,2,5, and 5) to the prime factors of 900 (2,2,3,3,5, and 5), we are missing prime factors 3, and 3. Since 3 x 3 = 9, the multiplier for 66/100 is 9.

$$\frac{66}{100} = \frac{66}{2 \times 2 \times 5 \times 5}$$

$$\frac{66}{100} = \frac{66 \times 9}{2 \times 2 \times 5 \times 5 \times 9} = \frac{594}{900}$$

3. Compare and order the numerators of each fraction.

$$\frac{594}{900} < \frac{675}{900} < \frac{700}{900} \qquad \text{SO} \qquad \frac{66}{100} < \frac{9}{12} < \frac{14}{18}$$

Converting Between Improper Fractions and Whole/Mixed Numbers

An improper fraction is one in which the numerator is larger than the denominator. For example, if you were told you had six-fourths (6/4) of a pie left, you would know that you had one whole pie (4/4) plus one-half of a pie (2/4).
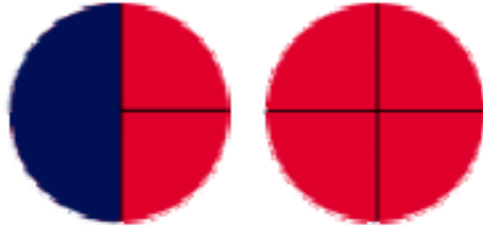
It can be more useful to express the fraction as a mixed number, a number containing a whole number and a fraction. Thus, rather than 6/4 of a pie, you say you have 1 1/2 pies.

To change an improper fraction into a mixed number:

1. Divide the denominator into the numerator.

2. Write the remainder as a fraction over the original denominator.

3. Reduce the remaining fraction.

Example

$$\frac{21}{9}$$

1. $9\overline{)21}$ = 2 remainder 3

2. $2\frac{3}{9}$

3. $2\frac{1}{3}$

Changing a Mixed Number into an Improper Fraction

When carrying out mathematical operations, it is usually necessary to work with

improper fractions rather than mixed numbers. To change a mixed number into an improper fraction:

1. Multiply the whole number by the denominator.

2. Add the numerator to the product.

3. Place the sum in the numerator over the original denominator.

Example

$$7\frac{2}{9} \qquad \textbf{1.} \quad 7 \times 9 = 63$$

$$\textbf{2.} \quad 63 + 2 = 65$$

$$\textbf{3.} \quad \frac{65}{9}$$

The general formula is:

$$a\frac{b}{c} = \frac{ac+b}{c}$$

Operations in fractions

Rule for Multiplication of Fractions

When multiplying fractions, simply multiply the numerators together and then multiply the denominators together.
Simplify the result.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

This works whether the denominators are the same or not.

- If you multiply the fractions 3/2 and 4/3 together, you get 12/6.

$$\frac{3}{2} \times \frac{4}{3} = \frac{3 \times 4}{2 \times 3} = \frac{12}{6}$$

- As with any solution, you should report the answer in simplified form. The fraction 12/6 simplifies to 2.

$$\frac{12}{6} = \frac{6 \times 2}{6} = \frac{\cancel{6} \times 2}{\cancel{6}} = 2$$

Example

What do you get when you multiply 1/2 and 3/7?

The result of multiplying these two fractions is 3/14.

$$\frac{1}{2} \times \frac{3}{7} = \frac{1 \times 3}{2 \times 7} = \frac{3}{14}$$

The fraction 3/14 cannot be simplified any further; it is in its simplest form.

Rule for Division of Fractions

When you divide two fractions, take the reciprocal of the second fraction and multiply. (Taking the reciprocal of a fraction means to flip it over.)

$$\frac{q}{r} \div \frac{s}{t} = \frac{q}{r} \times \frac{t}{s} = \frac{q \times t}{r \times s}$$

As with multiplication, this works whether the denominators are the same or not.

If you divide the fraction 3/2 by 4/3, you get the result shown below. As with any solution, you should report the answer in its simplified form. In this case, 9/8 is in its simplest form.

$$\frac{3}{2} \div \frac{4}{3} = \frac{3}{2} \times \frac{3}{4} = \frac{3 \times 3}{2 \times 4} = \frac{9}{8}$$

Example

What do you get when you divide 12/17 by 6/7?

We take the reciprocal of the second fraction and multiply it by the first. We get 82/102, which is not in its simplified form.

$$\frac{12}{17} \div \frac{6}{7} = \frac{12}{17} \times \frac{7}{6} = \frac{12 \times 7}{17 \times 6} = \frac{84}{102}$$

One easy way to simplify this fraction is to go back to the step before the numerator and denominator were multiplied.

$$\frac{12 \times 7}{17 \times 6}$$

To reduce a fraction to its simplest form, we need to find the prime factors of both the numerator and the denominator. When we do this for the numerator and denominator, we find we can cancel out a 2 and a 3 from the top and bottom.

$$\frac{12 \times 7}{17 \times 6} = \frac{2 \times 2 \times 3 \times 7}{17 \times 2 \times 3} = \frac{14}{17}$$

$$\frac{2 \times 7}{17} = \frac{14}{17}$$

Rule for Addition of Fractions
When adding fractions, make sure that the fractions being added have the same denominator. If they do not, find the LCD for the fractions and put each in its equivalent form. Then, simply add the numerators of the fractions.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

Rule for Subtraction of Fractions
When subtracting fractions, make sure that the fractions being subtracted have the same denominator. If they do not, find the LCD for the fractions and put each in its equivalent form. Then, simply subtract the numerators of the fractions.

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

The rules for adding and subtracting fractions can be broken down into several steps:

1. Determine whether the fractions have the same denominator. If the denominators are the same, move to step 4.
2. If the denominators are different, find the LCD for the fractions being added or subtracted.
3. For each of the original fractions, find its equivalent fraction with the LCD in the denominator.
4. Add or subtract the numerators of the fractions.
5. Simplify the resulting fraction.

Example of Adding Fractions

What is the sum of 3/4 and 1/3? Following the steps:

1. Determine whether the fractions have the same denominator. If the denominators are the same, move to step 4.

The two fractions do not have the same denominator. This means we need to find the LCD.

2. Find the LCD for the fractions being added.
a. Write the prime factors for the denominator of each fraction.
• _The prime factors of 4 are: 2 and 2.
• _The prime factor of 3 is: 3
b. Note all prime factors that occur. For each prime factor that occurs, determine in which denominator it occurs the most. Write down the prime factor the number of times it occurs in that one denominator. The prime factors that occur are 2, 2, and 3.
c. Calculate the LCD of your fractions. To do this, multiply the factors selected in step 2b.

<center>2 x 2 x 3= 12 (12 is our LCD)</center>

3. Find the equivalent fractions that have the LCD in the denominator.

Let's start with 3/4. Compare the prime factors of 4 (2 and 2) with the prime factors of 12 (2,2, and 3). The prime factor missing is a 3. Therefore, 3 is the multiplier for 3/4.

$$\frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}$$

For the fraction 1/3, compare the prime factors of 3 (3) with the prime factors of 12 (2,2, and 3). The prime factors that are missing are 2 and 2. Since 2 x 2= 4, 4 is the multiplier for the fraction 1/3.

$$\frac{1}{3} = \frac{1 \times 4}{3 \times 4} = \frac{4}{12}$$

4. Add the numerators of the fractions. Now that we have found the fractions that are equivalent to the ones we are adding, and these have the same denominator, we can add the fractions together.

$$\frac{3}{4} + \frac{1}{3} = \frac{9}{12} + \frac{4}{12} = \frac{13}{12}$$

5. Simplify the resulting fraction.

The answer of 13/12 is in its simplest form.

The steps for subtracting fractions are the same as for addition. The only difference is substituting subtraction for addition. If we wish to subtract 1/8 from 4/8, we can follow the steps outlined above.

1. Determine whether the fractions have the same denominator. If the denominators are the same, move to step 4.

The are the same, so we can move to step 4.

4. Subtract the numerators of the fractions.

$$\frac{4}{8} - \frac{1}{8} = \frac{4-1}{8} = \frac{3}{8}$$

5. Simplify the resulting fraction.

3/8 is in its simplest form.

As you can see, addition and subtraction of fractions is similar to adding and subtracting whole numbers. The important point is to be sure the fractions being added or subtracted have the same denominator.

NOTE: As explained above, the denominator of fractions must be in the same units (same denominator) to add them. We have shown how to find the LCD using prime factors. However, this process can be cumbersome.

Another way to get a common denominator is to multiply the denominators of the fractions you are adding. Then, multiply the numerator in one fraction by the denominator in the other fraction (and vice versa) to get the corresponding numerators.

That is, $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad}{bd} + \dfrac{bc}{bd} = \dfrac{ad + bc}{bd}$

Example

$$\frac{2}{3} + \frac{5}{7} = \frac{2(7) + 3(5)}{3(7)} = \frac{14 + 15}{21} = \frac{29}{21}$$

Equivalently,

$$\frac{2}{3} + \frac{5}{7} = \frac{2(7)}{3(7)} + \frac{3(5)}{3(7)} = \frac{14}{21} + \frac{15}{21} = \frac{29}{21}$$

In the example above, multiply 2 times 7 to get one numerator, and multiply 3 times 5 to get the other numerator. The denominator is 3 times 7.

When you are adding 3 fractions, multiply the numerator of one fraction by the denominators for the other two fractions. Do this for all three fractions. The common denominator is just the denominators for all three fractions multiplied together.

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf}$$

$$\frac{2}{3} + \frac{6}{5} + \frac{1}{2} = \frac{2 \times 5 \times 2}{30} + \frac{6 \times 3 \times 2}{30} + \frac{1 \times 3 \times 5}{30} = \frac{20}{30} + \frac{36}{30} + \frac{15}{30} = \frac{71}{30}$$